PART-III : MATHEMATICS

SECTION - 1

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+3	If ONLY the correct option is chosen;
Zero Marks	:	0	If none of the options is chosen (i.e. the question is unanswered);
Negative Marks	:	–1	In all other cases.

1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocentre of Δ is (1, 1), then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$

(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Answer (B)

Sol. As we know mirror image of orthocentre lie on circumcircle.

Image of (1, 1) in x-axis is (1, –1)

Image of (1, 1) in $x + y + 1 = 0$ is (–2, –2).

$\therefore$ The required circle will be passing through both (1, –1) and (–2, –2).

$\therefore$ Only $x^2 + y^2 + x + 3y = 0$ satisfy both.

2. The area of the region

$$\left\{(x, y) : 0 \le x \le \frac{9}{4}, \quad 0 \le y \le 1, \quad x \ge 3y, \quad x + y \ge 2\right\}$$

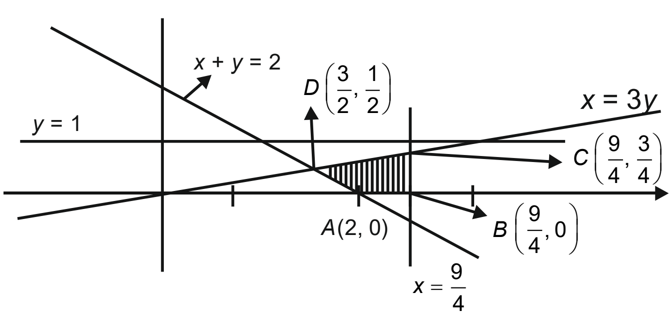
is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$

(C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Answer (A)

Sol. Rough sketch of required region is



∴ Required area is

Area of ΔACD + Area of ΔABC

i.e., $\frac{1}{4}+\frac{3}{32} = \frac{11}{32}$ sq. units

3. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 dentoe the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement from the set G_2 and let S_3 denote the set of these chosen elements.

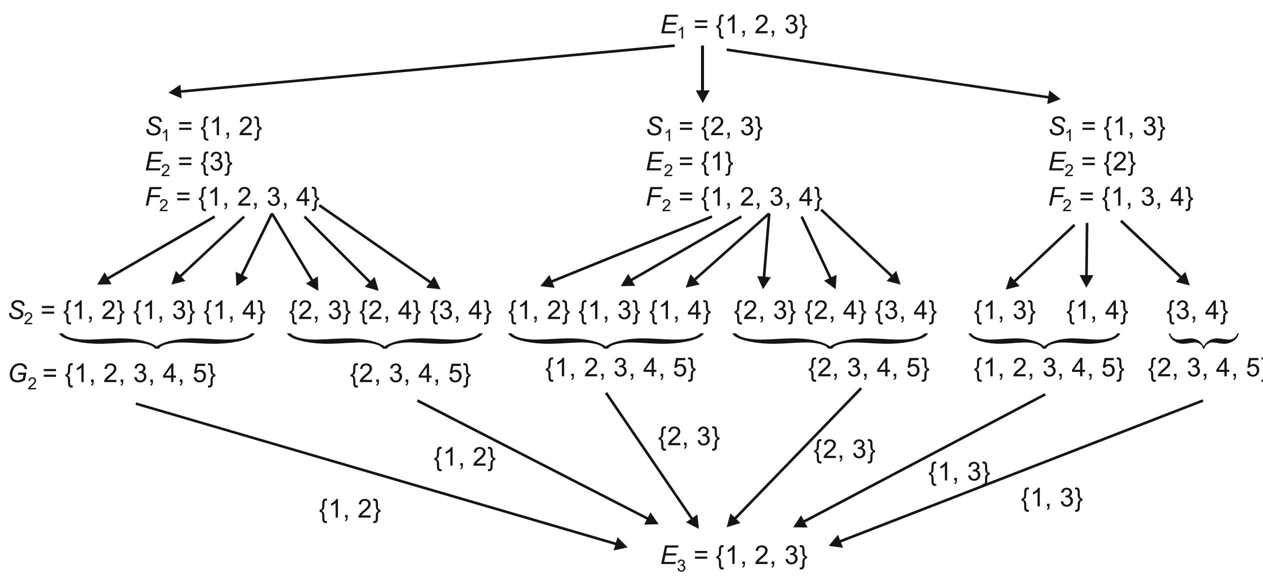
Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$

(C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Answer (A)

Sol. We will follow the tree diagram,



$$P(E_1 = E_3) = \frac{1}{3}\left[\frac{1}{2}\times\frac{1}{10}+\frac{1}{2}\times 0+\frac{1}{2}\times\frac{1}{10}+\frac{1}{2}\times\frac{1}{6}+\frac{2}{3}\times\frac{1}{10}+\frac{1}{3}\times 0\right]$$

$$= \frac{1}{3}\left[\frac{1}{4}\right]$$

$$\text{Required probability} = \frac{\frac{1}{3}\left[\frac{1}{2}\times\frac{1}{10}\right]}{\frac{1}{3}\times\frac{1}{4}} = \frac{1}{5}$$

4. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}, z_k = z_{k-1}e^{i\theta_k}$ for k = 2, 3, ..., 10, where $i = \sqrt{-1}$. Consider the statement P and Q given below:

$$P : |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$$

$$Q : |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$$

Then,

(A) P is **TRUE** and Q is **FALSE**

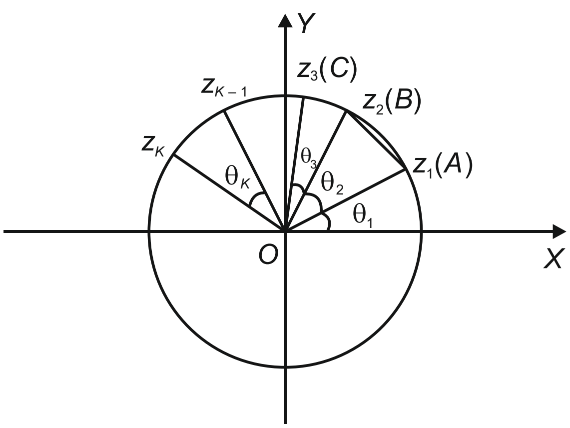
(B) Q is **TRUE** and P is **FALSE**

(C) Both P and Q are **TRUE**

(D) Both P and Q are **FALSE**

Answer (C)

Sol.



$|z_2 - z_1|$ = length of line $AB \le$ length of arc AB

$|z_3 - z_2|$ = length of line $BC \le$ length of arc BC

$\therefore$ Sum of length of these 10 lines $\le$ Sum of length of arcs (i.e. 2π)

(As $(\theta_1 + \theta_2 + ... + \theta_{10}) = 2\pi$)

$$\therefore \quad |z_2 - z_1| + |z_3 - z_2| + ... + |z_1 - z_{10}| \le 2\pi$$

And $|z_k^2 - z_{k-1}^2| = |z_k - z_{k-1}|\,|z_k + z_{k-1}|$

As we know $|z_k + z_{k-1}| \le |z_k| + |z_{k-1}| \le 2$

$$|z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_1^2 - z_{10}^2| \le 2\left(|z_2 - z_1| + |z_3 - z_2| + ... + |z_1 - z_{10}|\right)$$

$$\le 2\,(2\pi)$$

$\therefore$ Both (P) and (Q) are true.

SECTION - 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numerical keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+2	If ONLY the correct numerical value is entered at the designated place;
Zero Marks	:	0	In all other cases.

Question Stem for Question Nos. 5 and 6

Question Stem

Three numbers are chosen at random, one after another with replacement, from the set $S = \{1, 2, 3, ..., 100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

5. The value of $\frac{625}{4} p_1$ is _____.

Answer (76.25)

Sol. For p_1, we need to remove the cases when all three numbers are less than or equal to 80.

So, $p_1 = 1 - \left(\frac{80}{100}\right)^3 = \frac{61}{125}$

So, $\frac{625}{4} p_1 = \frac{625}{4} \times \frac{61}{125} = \frac{305}{4} = 76.25$

6. The value of $\frac{125}{4} p_2$ is _____.

Answer (24.50)

Sol. For p_2, we need to remove the cases when all three numbers are greater than 40.

So, $p_2 = 1 - \left(\frac{60}{100}\right)^3 = \frac{98}{125}$

So, $\frac{125}{4} p_2 = \frac{125}{4} \times \frac{98}{125} = 24.50$

Question Stem for Question Nos. 7 and 8

Question Stem

Let α, β and γ be real numbers such that the system of linear equations

$x + 2y + 3z = \alpha$

$4x + 5y + 6z = \beta$

$7x + 8y + 9z = \gamma - 1$

is consistent. Let $|M|$ represent the determinant of the matrix

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the **square** of the distance of the point (0, 1, 0) from the plane P.

7. The value of $|M|$ is _________.

Answer (1)

8. The value of D is _____.

Answer (1.50)

Sol. Solution for Q 7 and 8

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$$

Given system of equation will be consistent even if $\alpha = \beta = \gamma - 1 = 0$, i.e. equations will form homogeneous system.

So, $\alpha = 0$, $\beta = 0$, $\gamma = 1$

$$M = \begin{vmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = -1(-1) = +1$$

As given equations are consistent

$x + 2y + 3z - \alpha = 0$ ...P_1

$4x + 5y + 6z - \beta = 0$ ...P_2

$7x + 8y + 9z - (\gamma - 1) = 0$ ...P_3

For some scalar λ and μ

$\mu P_1 + \lambda P_2 = P_3$

$\mu(x + 2y + 3z - \alpha) + \lambda(4x + 5y + 6z - \beta) = 7x + 8y + 9z - (\gamma - 1)$

Comparing coefficients

$\mu + 4\lambda = 7$, $2\mu + 5\lambda = 8$, $3\mu + 6\lambda = 9$

$\lambda = 2$ and $\mu = -1$ satisfy all these conditions

comparing constant terms,

$-\alpha\mu - \beta\lambda = -(\gamma - 1)$

$\alpha - 2\beta + \gamma = 1$

So equation of plane is

$x - 2y + z = 1$

Distance from (0, 1, 0) = $\left|\frac{-2-1}{\sqrt{6}}\right| = \frac{3}{\sqrt{6}}$

$D = \left(\frac{3}{\sqrt{6}}\right)^2 = \frac{3}{2} = 1.50$

Question Stem for Question Nos. 9 and 10

Question Stem

Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ, let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S, where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'.

9. The value of λ^2 is _________.

Answer (9)

10. The value of D is _____.

Answer (77.14)

Sol. Solution for Q 9 and 10

$C : \left|\frac{x\sqrt{2} + y - 1}{\sqrt{3}}\right|\left|\frac{x\sqrt{2} - y + 1}{\sqrt{3}}\right| = \lambda^2$

$\Rightarrow C : \left|2x^2 - (y-1)^2\right| = 3\lambda^2$

C cuts $y - 1 = 2x$ at $R(x_1, y_1)$ and $S(x_2, y_2)$

So, $\left|2x^2 - 4x^2\right| = 3\lambda^2 \quad \Rightarrow \quad x = \pm\sqrt{\frac{3}{2}}\,|\lambda|$

So, $|x_1 - x_2| = \sqrt{6}\,|\lambda|$ and $|y_1 - y_2| = 2|x_1 - x_2| = 2\sqrt{6}\,|\lambda|$

$\because\ RS^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 \Rightarrow 270 = 30\lambda^2 \Rightarrow \lambda^2 = 9$

$\because$ Slope of RS = 2 and mid-point of RS is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \equiv (0, 1)$

So, $R'S' \equiv y - 1 = -\frac{1}{2}x$

Solving $y - 1 = -\frac{1}{2}x$ with 'C' we get $x^2 = \frac{12}{7}\lambda^2$

$\Rightarrow |x_1 - x_2| = 2\sqrt{\frac{12}{7}}\,|\lambda|$ and $|y_1 - y_2| = \frac{1}{2}|x_1 - x_2| = \sqrt{\frac{12}{7}}\,|\lambda|$

Hence, $D = (R'S')^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = \frac{12}{7}.9 \times 5 \approx 77.14$

SECTION - 3

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	:	+4	If only (all) the correct option(s) is(are) chosen;
Partial Marks	:	+3	If all the four options are correct but ONLY three options are chosen;
Partial Marks	:	+2	If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	:	+1	If two or more options are correct but ONLY one option is chosen and it is a correct option;
Zero Marks	:	0	If unanswere;
Negative Marks	:	–2	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2 marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e., the question is unanswered) will get 0 marks and

  choosing any other option(s) will get –2 marks.

11. For any 3 × 3 matrix M, let $|M|$ denote the determinant of M. Let

$$E=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 8 & 13 & 18\end{bmatrix},\ P=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \text{ and } F=\begin{bmatrix}1 & 3 & 2\\ 8 & 18 & 13\\ 2 & 4 & 3\end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is(are) **TRUE**?

(A) $F = PEP$ and $P^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(B) $\left|EQ+PFQ^{-1}\right|=\left|EQ\right|+\left|PFQ^{-1}\right|$

(C) $\left|(EF)^3\right|>\left|EF\right|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Answer (A, B, D)

Sol. $\because$ P is formed from I by exchanging second and third row or by exchanging second and third column.

So, PA is a matrix formed from A by changing second and third row.

Similarly AP is a matrix formed from A by changing second and third column.

Hence, Tr(PAP) = Tr(A) ...(1)

(A) Clearly, $P.P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$

and $PE = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{bmatrix} \Rightarrow PEP = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix} = F$

$\Rightarrow$ $PEP = F \Rightarrow PFP = E$ ...(2)

(B) $\because |E| = |F| = 0$

So, $\left|EQ + PFQ^{-1}\right| = \left|PFPQ + PFQ^{-1}\right| = |P|\,|F|\,\left|PQ + Q^{-1}\right| = 0$

Also, $|EQ| + \left|PFQ^{-1}\right| = 0$

(C) From (2); $PFP = E$ and $|P| = -1$

So, $|F| = |E|$

Also, $|E| = 0 = |F|$

So, $|EF|^3 = 0 = |EF|^2$

(D) $\because P^2 = I \quad \Rightarrow \quad P^{-1} = P$

So, $\text{Tr}\left(P^{-1}EP + F\right) = \text{Tr}(PEP + F) = \text{Tr}(2F)$

Also $\text{Tr}\left(E + P^{-1}FP\right) = \text{Tr}(E + PFP) = \text{Tr}(2E)$

Given that Tr(E) = Tr(F)

$\Rightarrow$ Tr(2E) = Tr(2F)

12. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$$

Then which of the following statements is (are) **TRUE**?

(A) f is decreasing in the interval (–2, –1)

(B) f is increasing in the interval (1, 2)

(C) f is onto

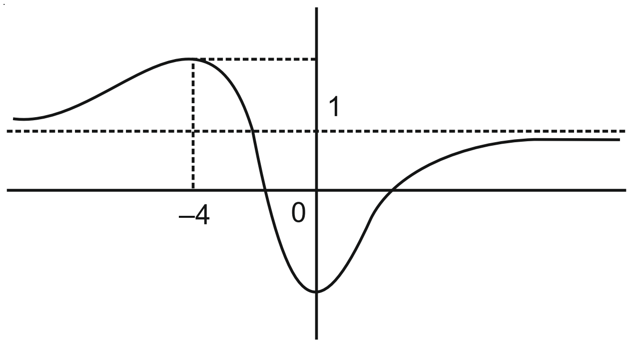
(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

Answer (A, B)

Sol. $f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$\Rightarrow \quad f'(x) = \dfrac{5x(x+4)}{(x^2 + 2x + 4)^2}$

$\Rightarrow$ $f(x)$ has local maxima at $x = -4$ and minima at $x = 0$



Range of $f(x)$ is $\left[-\frac{3}{2}, \frac{11}{6}\right]$

13. Let E, F and G be three events having probabilities

$P(E) = \frac{1}{8}, P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$, and $P(E \cap F \cap G) = \frac{1}{10}$.

For any event H, if H^c denotes its complement, then which of the following statements is(are) **TRUE**?

(A) $P(E \cap F \cap G^c) \le \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \le \frac{1}{15}$

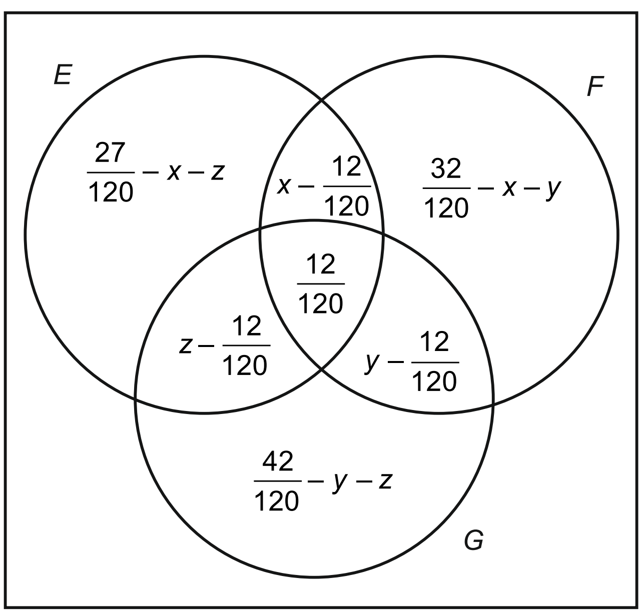
(C) $P(E \cup F \cup G) \le \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \le \frac{5}{12}$

Answer (A, B, C)

Sol. Let $P(E \cap F) = x$, $P(F \cap G) = y$ and $P(E \cap G) = z$

Clearly $x, y, z \ge \frac{1}{10}$



$\because x + z \le \frac{27}{120} \Rightarrow x, z \le \frac{15}{120}$

$x + y \le \frac{32}{120} \Rightarrow x, y \le \frac{20}{120}$

and $y+z \le \frac{42}{120} \Rightarrow y, z \le \frac{30}{120}$

Now $P(E \cap F \cap G^c) = x - \frac{12}{120} \le \frac{3}{120} = \frac{1}{40}$

$P(E^c \cap F \cap G) = y - \frac{12}{120} \le \frac{80}{120} = \frac{1}{15}$

$P(E \cup F \cup G) \le P(E) + P(F) + P(G) = \frac{13}{24}$

and $P(E^c \cap F^c \cap G^c) = 1 - P(E \cup F \cup G) \ge \frac{11}{24} \ge \frac{5}{12}$

14. For any 3 × 3 matrix *M*, let |*M*| denote the determinant of *M*. Let *I* be the 3 × 3 identify matrix. Let *E* and *F* be two 3 × 3 matrices such that (*I* – *EF*) is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) **TRUE**?

(A) $|FE| = |I - FE|\,|FGE|$ (B) $(I - FE)(I + FGE) = I$

(C) $EFG = GEF$ (D) $(I - FE)(I - FGE) = I$

Answer (A, B, C)

Sol. $\because\ I - EF = G^{-1}$

$\Rightarrow$ G – GEF = I …(1)

and $G - EFG = I$ …(2)

Clearly $GEF = EFG$ (option C is correct)

Also $(I - FE)(I + FGE) = I - FE + FGE - FE + FGE$

$= I - FE + FGE - F(G - I)E$

$= I - FE + FGE - FGE + FE$

$= I$ (option B is correct and D is incorrect)

Now, $(I - FE)(I - FGE) = I - FE - FGE + F(G - I)E$

$= I - 2FE$

$\Rightarrow (I - FE)(-FGE) = -FE$

$\Rightarrow |I - FE||FGE| = |FE|$

15. For any positive integer *n*, let $S_n : (0, \infty) \to \mathbb{R}$ be defined by

$$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1 + k(k+1)x^2}{x}\right)$$

where for any $x \in \mathbb{R}$, $\cot^{-1}(x) \in (0, \pi)$ and $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then which of the following statements is (are) **TRUE**?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1 + 11x^2}{10x}\right)$, for all $x > 0$

(B) $\lim\limits_{n \to \infty} \cot(S_n(x)) = x$, for all $x > 0$

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and $x > 0$

Answer (A, B)

Sol. $S_n(x) = \sum_{k=1}^{n} \tan^{-1}\left(\frac{(k+1)x - kx}{1 + kx.(k+1)x}\right)$

$$= \sum_{k=1}^{n}\left(\tan^{-1}(k+1)x - \tan^{-1}kx\right)$$

$$= \tan^{-1}(n+1)x - \tan^{-1}x = \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$$

Now (A) $S_{10}(x) = \tan^{-1}\left(\frac{10x}{1+11x^2}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$

(B) $\lim_{n\to\infty} \cot(S_n(x)) = \cot\left(\tan^{-1}\left(\frac{x}{x^2}\right)\right) = x$

(C) $S_3(x) = \frac{\pi}{4}$ $\Rightarrow$ $\frac{3x}{1+4x^2} = 1$ $\Rightarrow$ $4x^2 - 3x + 1 = 0$ has no real root.

(D) For $x = 1$, $\tan(S_n(x)) = \frac{n}{n+2}$ which is greater than $\frac{1}{2}$ for $n \geq 3$ so this option is incorrect.

16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$, where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$x^2 + y^2 + 5x - 3y + 4 = 0$

Then which of the following statements is (are) **TRUE**?

(A) $\alpha = -1$

(B) $\alpha\beta = 4$

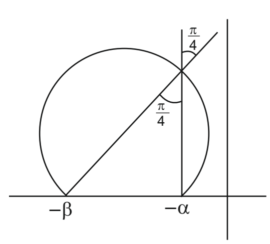
(C) $\alpha\beta = -4$

(D) $\beta = 4$

Answer (B, D)

Sol. Circle $x^2 + y^2 + 5x - 3y + 4 = 0$ cuts the real axis (x-axis) at (–4, 0), (–1, 0)

Clearly $\alpha = 1$ and $\beta = 4$



SECTION - 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the moust and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered.

Zero Marks : 0 In all other cases.

17. For $x \in \mathbb{R}$, the number of real roots of the equation

$3x^2 - 4|x^2 - 1| + x - 1 = 0$

is ________.

Answer (4)

Sol. $3x^2 - 4\left|x^2 - 1\right| + x - 1 = 0$

Let $x \in [-1, \ 1]$

$\Rightarrow \ 3x^2 - 4(-x^2 + 1) + x - 1 = 0$

$\Rightarrow \ 3x^2 + 4x^2 - 4 + x - 1 = 0$

$\Rightarrow \ 7x^2 + x - 5 = 0$

$\Rightarrow \ x = \dfrac{-1 \pm \sqrt{1 + 140}}{2}$

Both values acceptable

Let $x \in (-\infty, -1) \cup (1, \ \infty)$

$x^2 - 4(x^2 - 1) + x - 1 = 0$

$\Rightarrow \ x^2 - x - 3 = 0$

$\Rightarrow \ x = \dfrac{1 \pm \sqrt{1 + 12}}{2}$

Again both are acceptable

Hence total number of solution = 4

18. In a triangle ABC, let $AB = \sqrt{23}$, and BC = 3 and CA = 4. Then the value of

$$\frac{\cot A + \cot C}{\cot B}$$

is ______.

Answer (2)

Sol. With standard notations

Given : $c=\sqrt{23}$, $a = 3$, $b = 4$

Now $\dfrac{\cot A+\cot C}{\cot B}=\dfrac{\dfrac{\cos A}{\sin A}+\dfrac{\cos C}{\sin C}}{\dfrac{\cos B}{\sin B}}$

$$=\frac{\dfrac{b^2+c^2-a^2}{2bc.\sin A}+\dfrac{a^2+b^2-c^2}{2ab\sin C}}{\dfrac{c^2+a^2-b^2}{2ac\sin B}}$$

$$=\frac{\dfrac{b^2+c^2-a^2}{4\Delta}+\dfrac{a^2+b^2-c^2}{4\Delta}}{\dfrac{c^2+a^2-b^2}{4\Delta}}=\frac{2b^2}{a^2+c^2-b^2}=2$$

19. Let $\vec{u},\vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and

$\vec{u}\cdot\vec{w}=1,\quad \vec{v}\cdot\vec{w}=1,\quad \vec{w}\cdot\vec{w}=4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u},\vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u}+5\vec{v}|$ is ________.

Answer (7)

Sol. Given $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]=\sqrt{2}$

Also $\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]^2=\begin{vmatrix}\vec{u}.\vec{u} & \vec{u}\cdot\vec{v} & \vec{u}\cdot\vec{w}\\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w}\\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{v} & \vec{w}\cdot\vec{w}\end{vmatrix}=2$

Let $\vec{u}.\vec{v}=k$ and substitute rest values, we get

$$\begin{vmatrix}1 & K & 1\\ K & 1 & 1\\ 1 & 1 & 4\end{vmatrix}=2$$

$\Rightarrow$ $4K^2 - 2K = 0$

$\Rightarrow$ $\vec{u}.\vec{v}=0$ (rejected) or $\vec{u}.\vec{v}=\dfrac{1}{2}$

$\therefore$ $\vec{u}.\vec{v}=\dfrac{1}{2}$

$|3\vec{u}+5\vec{v}|^2=9+25+30\times\dfrac{1}{2}=49$

$\Rightarrow$ $|3\vec{u}+5\vec{v}|=7$