CBSE Paper 2020 Math Delhi (Set 2)

General Instructions:

Read the following instructions very carefully and strictly follow them :
(i) This question paper comprises **four** sections – **A, B, C** and **D.**
This question paper carries **36** questions. **All** questions are compulsory.
(ii) **Section A** – Question no. **1** to **20** comprises of **20** questions of **one** mark each.
(iii) **Section B** – Question no. **21** to **26** comprises of **6** questions of **two** marks each.
(iv) **Section C** – Question no. **27** to **32** comprises of **6** questions of **four** marks each.
(v) **Section D** – Question no. **33** to **36** comprises of **4** questions of **six** marks each.
(vi) There is no overall choice in the question paper. However, an internal choice has been provided in **3** questions of one mark, **2** questions of two marks, **2** questions of four marks and **2** questions of six marks. Only one of the choices in such questions have to be attempted.
(vii) In addition to this, separate instructions are given with each section and question, wherever necessary.
(viii) Use of calculators is not permitted.

Question 1

If $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = 0$, then x equals

(a) 0
(b) −2
(c) −1
(d) 2

Solution:

Given: $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = \mathrm{O}$

$\Rightarrow \begin{bmatrix} x-2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$\Rightarrow x = 2$

Hence, the correct answer is option (d).

Question 2

$\int 4^x 3^x \, dx$ equals

(a) $\frac{12^x}{\log 12} + \mathrm{C}$

(b) $\frac{4^x}{\log 4}+\mathrm{C}$

(c) $\left(\frac{4^x \cdot 3^x}{\log 4 \cdot \log 3}\right)+\mathrm{C}$

(d) $\frac{3^x}{\log 3}+\mathrm{C}$

Solution:

Consider the given integral

$\int 4^x 3^x \mathrm{d}x$

$=\int 12^x \mathrm{d}x$

$=\frac{1}{\ln 12} \int\left(12^x \ln 12\right) \mathrm{d}x$

$=\frac{12^x}{\ln 12}+C$

Hence, the correct answer is option (a).

Question 3

A number is chosen randomly from numbers 1 to 60. The probability that the chosen number is a multiple of 2 or 5 is

(a) 2/5

(b) 3/5

(c) 7/10

(d) 9/10

Solution:

The number of multiples of 2 from 1 to 60 are given by,

$60=2+\left(n_1-1\right)2$

$\Rightarrow 2\left(n_1-1\right)=58$

$\Rightarrow n_1=30$

The number of multiples of 5 from 1 to 60 are given by,

$60=5+\left(n_2-1\right)5$

$\Rightarrow 5\left(n_2-1\right)=55$

$\Rightarrow n_1=12$

The number of multiples of both 2 and 5 from 1 to 60 are given by,

$60=10+\left(n_3-1\right)10$

$\Rightarrow 10\left(n_3-1\right)=50$

$\Rightarrow n_3=6$

Therefore,

$$\text{Required probability} = \frac{n_1+n_2-n_3}{60}$$
$$= \frac{30+12-6}{60}$$
$$= \frac{36}{60}$$
$$= \frac{3}{5}$$

Hence, the correct answer is option (b).

Question 4

ABCD is a rhombus whose diagonals intersect at E. Then $\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$ equals

(a) $\overrightarrow{0}$

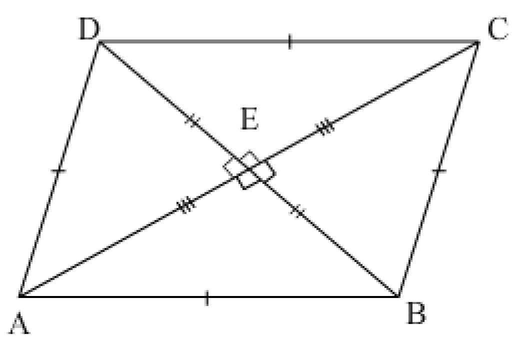
(b) $\overrightarrow{AD}$

(c) $2\overrightarrow{BC}$

(d) $2\overrightarrow{AD}$

Solution:

Given: ABCD is a rhombus and its diagonal intersects at E.



According to the properties of rhombus,

$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right| = \left|\overrightarrow{CD}\right| = \left|\overrightarrow{DA}\right|$$
$$\overrightarrow{EA} = -\overrightarrow{EC}$$
$$\overrightarrow{ED} = -\overrightarrow{EB}$$

Consider the given expression

$$\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$$

$$=-\overrightarrow{EC}+\overrightarrow{EB}+\overrightarrow{EC}-\overrightarrow{EB} \qquad \left(\because \overrightarrow{EA}=-\overrightarrow{EC} \text{ and } \overrightarrow{ED}=-\overrightarrow{EB}\right)$$

$$=0$$

Hence, the correct answer is option (a).

Question 5

A card is picked at random from a pack of 52 playing cards. Given that picked card is a queen, the probability of this card to be a card of spade is

(a) 1/3
(b) 4/13
(c) 1/4
(d) 1/2

Solution:

Given that the picked card is a queen.
Therefore,

$$\text{Total number of outcomes}=\text{Total number of queens}=4$$

Since only one queen exists of spade.
Therefore,

$$\text{Required Probability}=\frac{\text{Total number of favorable outcomes}}{\text{Total outcomes}}=\frac{1}{4}$$

Hence, the correct answer is option (c).

Question 6

If $\hat{i},\ \hat{j},\ \hat{k}$ are unit vectors along three mutually perpendicular directions, then

(a) $\hat{i}\ .\ \hat{j}=1$
(b) $\hat{i}\ \times\ \hat{j}=1$
(c) $\hat{i}\ .\ \hat{k}=0$
(d) $\hat{i}\ \times\ \hat{k}=0$

Solution:

Given three unit vectors $\hat{i},\ \hat{j},\ \hat{k}$ are mutually perpendicular.
Therefore, the angle between them will be right angle.
Consider

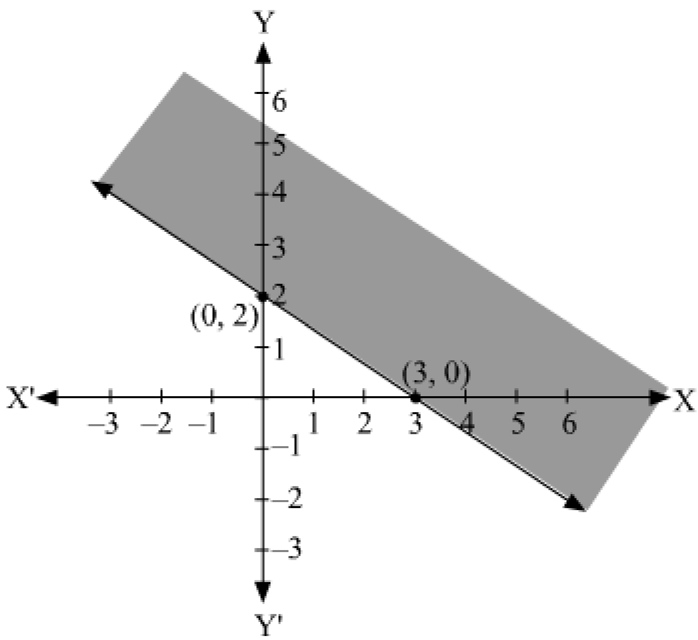
$$\hat{i}\cdot\hat{k}=\left|\hat{i}\right|\left|\hat{k}\right|\cos 90^\circ=0$$

Hence, the correct answer is option (c).

Question 7

The graph of the inequality $2x + 3y > 6$ is
(a) half plane that contains the origin.
(b) half plane that neither contains the origin nor the points of the line $2x + 3y = 6$.
(c) whole XOY – plane excluding the points on the line $2x + 3y = 6$.
(d) entire XOY plane.

Solution:



From the graph it can be clearly seen that the graph of the inequality $2x + 3y > 6$ is a half plane that neither contains the origin nor the points of the line $2x + 3y = 6$.
Hence, the correct answer is option (b).

Question 8

If A is a square matrix of order 3, such that A (adj A) = 10 I, then |adj A| is equal to
(a) 1
(b) 10
(c) 100
(d) 101

Solution:

We know that $(adjA)A=|A|I$
Writing determinants on both the sides, we get
$|(adjA)A|=||A|I|$
$\Rightarrow |10I|=|A|$
$\Rightarrow |A|=|10I|=10$

Also,
$|adjA|=|A|^{n-1}$
$\Rightarrow |adjA|=|10|^{3-1}=100$

Hence, the correct answer is option (c).

Question 9

The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$ are mutually perpendicular if the value of k is

(a) $-\frac{2}{3}$

(b) $\frac{2}{3}$

(c) −2

(d) 2

Solution:

The given two lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{4-z}{k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{-2}$ are mutually perpendicular.
Their direction ratios are $(1,\ 1,\ -k)$ and $(k,\ 2,\ -2)$ respectively.
$\Rightarrow 1\left(k\right)+1\left(2\right)-k\left(-2\right)=0$
$\Rightarrow k+2+2k=0$
$\Rightarrow 3k+2=0$
$\Rightarrow k=\frac{-2}{3}$
Hence, the correct answer is option (a).

Question 10

If y = Ae^{5x} + Be^{-5x}, then $\frac{d^2y}{dx^2}$ is equal to
(a) 25y
(b) 5y
(c) −25y
(d) 15y

Solution:

Given $y = Ae^{5x} + Be^{-5x}$.
Differentiating both sides w.r.t. *x*, we get

$$\frac{\mathrm{d}\,y}{\mathrm{d}\,x} = 5Ae^{5x} - 5Be^{-5x}$$

$$\Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 25Ae^{5x} + 25Be^{-5x}$$

$$\Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}\,x^2} = 25\left(Ae^{5x} + Be^{-5x}\right) = 25y$$

Hence, the correct answer is option (a).

Question 11

Fill in the blank.
A relation R on a set A is called ________, if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$ implies that $(a_1, a_3) \in R$, for $a_1, a_2, a_3 \in A$.

Solution:

If $(a_1, a_2) \in \mathrm{R}$ and $(a_2, a_3) \in \mathrm{R}$ then $(a_1, a_3) \in \mathrm{R}$
This category of relation is called transitive.
Hence, set A is called a transitive set.

Question 12

Fill in the blank.
The integrating factor of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is ________.

OR

Fill in the blank.
The degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = x$ is ____________.

Solution:

The given differential equation is

$$x\frac{dy}{dx} + 2y = x^2$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2}{x}\right)y = x$$

Which is of the form $\frac{dy}{dx} + Py = Q$

Hence, the integrating factor is $e^{\int P dx} = e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

OR

The given differential equation is $1 + \left(\frac{dy}{dx}\right)^2 = x$.

Since the highest order derivative involved in the given differential equation is $\frac{dy}{dx}$ and its power is 2.
So, the degree of the given differential equation is 2.

Question 13

Fill in the blank.

The vector equation of a line which passes through the points (3, 4, −7) and (1, −1, 6) is ________.

OR

Fill in the blank.

The line of shortest distance between two skew lines is _____ to both the lines.

Solution:

Let the two vectors be $\vec{a}=3\hat{i}+4\hat{j}-7\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+6\hat{k}$.

Now, the vector equation of a line passing through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is given by

$$\begin{aligned}\vec{r}&=\vec{a}+\lambda\left(\vec{b}-\vec{a}\right)\\&=\left(3\hat{i}+4\hat{j}-7\hat{k}\right)+\lambda\left[\left(\hat{i}-\hat{j}+6\hat{k}\right)-\left(3\hat{i}+4\hat{j}-7\hat{k}\right)\right]\\&=\left(3\hat{i}+4\hat{j}-7\hat{k}\right)+\lambda\left(-2\hat{i}-5\hat{j}+13\hat{k}\right)\\&=(3-2\lambda)\hat{i}+(4-5\lambda)\hat{j}+(-7+13\lambda)\hat{k}\end{aligned}$$

OR

The line of shortest distance between two skew lines is perpendicular to both the lines.

Question 14

If $\mathrm{A}+\mathrm{B}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $\mathrm{A}-2\mathrm{B}=\begin{bmatrix}-1&1\\0&-1\end{bmatrix}$, then A = ________.

Solution:

Given that

$$A+B=\begin{bmatrix}1&0\\1&1\end{bmatrix} \qquad \ldots\ldots(1)$$

$$\text{and }\mathrm{A}-2\mathrm{B}=\begin{bmatrix}-1&1\\0&-1\end{bmatrix} \qquad \ldots\ldots(2)$$

Multiply (1) with 2 and add with (2), we get

$$2A+A=\begin{bmatrix}2&0\\2&2\end{bmatrix}+\begin{bmatrix}-1&1\\0&-1\end{bmatrix}$$

$$\Rightarrow 3A=\begin{bmatrix}1&1\\2&1\end{bmatrix}$$

$$\Rightarrow A=\tfrac{1}{3}\begin{bmatrix}1&1\\2&1\end{bmatrix}$$

Question 15

Fill in the blank.

The least value of the function $f(x) = ax + \frac{b}{x}\,(a > 0,\ b > 0,\ x > 0)$ is ________.

Solution:

Given: $f(x) = ax + \frac{b}{x}\,(a > 0,\ b > 0,\ x > 0)$

As $a,\ b$ and $x > 0$

We can use $\mathrm{AM} \geq \mathrm{GM}$

$$\frac{ax+\frac{b}{x}}{2} \geq \sqrt{ax \times \frac{b}{x}}$$

$$\frac{ax+\frac{b}{x}}{2} \geq \sqrt{ab}$$

$$ax + \frac{b}{x} \geq 2\sqrt{ab}$$

Hence, the minimum value of $f(x)$ is $2\sqrt{ab}$.

Question 16

Evaluate : $\sin\ \left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$.

Solution:

Consider the given expression

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} - \left(\frac{-\pi}{6}\right)\right] \qquad \left[\because \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}\right]$$

$$= \sin\frac{\pi}{2}$$

$$= 1$$

Question 17

Using differential, find the approximate value of $\sqrt{36.6}$ upto 2 decimal places.

OR

Find the slope of tangent to the curve $y = 2\cos^2(3x)$ at $x = \frac{\pi}{6}$.

Solution:

Consider the given expression

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(\frac{-1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}-\left(\frac{-\pi}{6}\right)\right] \qquad \left[\because \sin^{-1}\left(\frac{-1}{2}\right)=\frac{-\pi}{6}\right]$$

$$=\sin\frac{\pi}{2}$$

$$=1$$

Question 18

Find the value of $\int\limits_1^4 |x-5|dx.$

Solution:

$$I=\int_1^4 |x-5|dx$$

$$I=\int_1^4 -(x-5)dx$$

$$I=\int_1^4 (5\ -\ x)dx$$

$$I=\left[5x\ -\ \frac{x^2}{2}\right]_1^4$$

$$I=\left(5\times 4\ -\ \frac{4^2}{2}\right)-\left(5\times 1-\frac{1^2}{2}\right)$$

$$I=12\ -\ \frac{9}{2}=\frac{15}{2}$$

Question 19

If the function *f* defined as

$$f(x)=\begin{cases}\frac{x^2-9}{x-3}, & x\neq 3\\ k, & x=3\end{cases}$$

is continuous at $x = 3$, find the value of k.

Solution:

The function is defined as $f(x)=\begin{cases}\frac{x^2-9}{x-3},\ x\neq 3\\ k,\ x=3\end{cases}$

This function is given to be continuous at $x = 3$.

$$\Rightarrow \lim_{x\to 3^+} f(x)=f(3)$$

$\Rightarrow \lim_{x \to 3^+} \left[\frac{x^2-9}{x-3}\right] = k$

$\Rightarrow \lim_{x \to 3^+} \left[\frac{(x-3)(x+3)}{x-3}\right] = k$

$\Rightarrow \lim_{x \to 3^+} (x+3) = k$

$\Rightarrow 3+3 = k$

$\Rightarrow k = 6$

Question 20

For $\mathrm{A} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ write A^{-1}.

Solution:

Given $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -4 \\ 1 & -1 \end{vmatrix} = 1 \neq 0$

This implies that the given matrix is invertible.

Also, $adjA = \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$

$\therefore A^{-1} = \frac{adjA}{|A|} = \begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$

Question 21

Find $\int \frac{x+1}{x(1-2x)} dx.$

Solution:

Converting the given integrand into partial fractions as follows

$\frac{x+1}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}$

$\Rightarrow x+1 = A(1-2x) + Bx$

$\Rightarrow x+1 = (-2A+B)x + A$

$\Rightarrow A = 1,\ B = 3$

$\int \frac{x+1}{x(1-2x)} \mathrm{d}x = \int \frac{1}{x} \mathrm{d}x + \int \frac{3}{1-2x} \mathrm{d}x$

$= \ln x + \frac{3}{2} \ln \left(x - \frac{1}{2}\right) + C$

Question 22

Evaluate $\int \frac{x \ \sin^{-1}(x^2)}{\sqrt{1-x^4}} dx.$

Solution:

Consider the given integral

$\int \frac{x \sin^{-1}(x^2)}{\sqrt{1-x^4}} \mathrm{d}x$

Let $\sin^{-1}(x^2) = t$

$\Rightarrow \frac{x}{\sqrt{1-x^4}} \mathrm{d}x = \frac{\mathrm{d}t}{2}$

Therefore,

$\int \frac{\mathrm{x}\sin^{-1}(\mathrm{x}^2)}{\sqrt{1-\mathrm{x}^4}} \mathrm{dx} = \frac{1}{2}\int t \mathrm{d}t = \frac{t^2}{4} + C$

$\Rightarrow \int \frac{\mathrm{x}\sin^{-1}(\mathrm{x}^2)}{\sqrt{1-\mathrm{x}^4}} \mathrm{dx} = \frac{[\sin^{-1}(x^2)]^2}{4} + C$

Question 23

Find the value of $\int\limits_0^1 x(1-x)^n \ dx.$

Solution:

The given integral is $\int_0^1 x(1-x)^n \, \mathrm{d}\, x$

Put $1 - x = t$ so that $\mathrm{d}x = -\mathrm{d}t$

$\Rightarrow t = 1$ when $x = 0$, $t = 0$ when $x = 1$

$\Rightarrow -\int_1^0 (1-t)t^n \, \mathrm{d}\, t$

$\Rightarrow -\int_1^0 \left(t^n - t^{n+1}\right) \mathrm{d}\, t$

$\Rightarrow \int_0^1 \left(t^n - t^{n+1}\right) \mathrm{d}\, t$

$\Rightarrow \left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2}\right]_0^1$

$\Rightarrow \left[\frac{(1)^{n+1}}{n+1} - \frac{(1)^{n+2}}{n+2}\right] - \left[\frac{(0)^{n+1}}{n+1} - \frac{(0)^{n+2}}{n+2}\right]$

$\Rightarrow \left[\frac{1}{n+1} - \frac{1}{n+2}\right]$

$\Rightarrow \frac{1}{(n+1)(n+2)}$

Question 24

If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$.

Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

Solution:

Given $x = a\cos\theta,\ y = b\sin\theta$
Differentiating w.r.t. θ, we get
$\frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = b\cos\theta$
$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = \frac{-b}{a}\cot\theta$

Differentiating w.r.t. x, we get
$\frac{d^2y}{dx^2} = \frac{b\operatorname{cosec}^2\theta}{a}\ \frac{d\theta}{dx}$
$= \frac{b\operatorname{cosec}^2\theta}{a}\left(\frac{-1}{a\sin\theta}\right)$
$= \frac{-b\operatorname{cosec}^3\theta}{a^2}$

OR

Let $y = \sin^2 x,\ z = e^{\cos x}$
Differentiating both the functions w.r.t. x, we get
$\frac{dy}{dx} = 2\sin x\cos x,\ \frac{dz}{dx} = e^{\cos x}(-\sin x)$
$\Rightarrow \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2\sin x\cos x}{e^{\cos x}(-\sin x)} = \frac{-2\cos x}{e^{\cos x}}$

Question 25

Given two independent events A and B such that P(A) = 0.3 and P(B) = 0.6, find P(A′ ∩ B′)

Solution:

It is given that P (A) = 0.3 and P (B) = 0.6
Also, A and B are independent events.

$\therefore P(A \text{ and } B) = P(A) \cdot P(B)$
$\Rightarrow P(A \cap B) = 0.3 \times 0.6 = 0.18$
And, $P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$= 0.3 + 0.6 - 0.18$
$= 0.72$
So, $P(A' \cap B')$
$= P\left((A \cup B)'\right)$
$= 1 - P(A \cup B)$
$= 1 - 0.72$
$= 0.28$

Question 26

If $f(x) = \frac{4x+3}{6x-4},\ x \neq \frac{2}{3}$, then show that (*fof*) (*x*) = *x*, for all $x \neq \frac{2}{3}$. Also, write inverse of *f*.

OR

Check if the relation *R* in the set $\mathbb{R}$ of real numbers defined as *R* = {(*a*, *b*) : *a* < *b*} is (i) symmetric, (ii) transitive

Solution:

Given, $f(x) = \frac{4x+3}{6x-4},\ x \neq \frac{2}{3}$

$$(fof)(x) = \frac{4\left(\frac{4x+3}{6x-4}\right)+3}{6\left(\frac{4x+3}{6x-4}\right)-4}$$

$$\Rightarrow (fof)(x) = \frac{\frac{16x+12+18x-12}{6x-4}}{\frac{24x+18-24x+16}{6x-4}}$$

$$\Rightarrow (fof)(x) = \frac{34x}{34} = x$$

Hence proved.

To find inverse of the given function,
Let $y = \frac{4x+3}{6x-4}$
$\Rightarrow y(6x-4) = 4x+3$

$\Rightarrow 6xy - 4y = 4x + 3$
$\Rightarrow 6xy - 4x = 3 + 4y$
$\Rightarrow x(6y - 4) = 4y + 3$
$\Rightarrow x = \frac{4y+3}{6y-4}$
Therefore, $f^{-1}(x) = \frac{4x+3}{6x-4},\ x \neq \frac{2}{3}$

OR

We have,
$R = \{(a,\ b)\ :\ a < b\}, \quad \text{where } a,\ b \in \mathbb{R}$
(i) Symmetry
We observe that $(2,\ 3) \in R$ but $(3,\ 2) \notin R$.
So, *R* is not symmetric.

(ii) Transitivity
Let $(a,\ b) \in R$ and $(b,\ c) \in R$. Then,

$\Rightarrow a < b$ and $b < c$
$\Rightarrow a < c$
$\Rightarrow (a, c) \in R$
So, *R* is transitive.

Question 27

Prove that $\tan\ \left[2\ \tan^{-1}\left(\frac{1}{2}\right) - \cos^{-1}\ 3\right] = \frac{9}{13}$.

Solution:

Taking the LHS
$\tan\left[2\tan^{-1}\left(\frac{1}{2}\right) - \cot^{-1} 3\right]$

$$= \tan\left[\tan^{-1}\frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} - \tan^{-1}\left(\frac{1}{3}\right)\right] \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{4}{3}\right) - \tan^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \tan\left[\tan^{-1}\frac{\left(\frac{4}{3}\right)-\left(\frac{1}{3}\right)}{1+\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)}\right] \qquad \left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right)$$

$$= \tan\left[\tan^{-1}\left(\frac{9}{13}\right)\right] \qquad \left[\because \tan\left(\tan^{-1}x\right) = x\right]$$

$$= \frac{9}{13}$$

LHS=RHS

Question 28

If $y=(\cos\ x)^{x}+\tan^{-1}\sqrt{x},\ \text{find}\ \frac{dy}{dx}.$

Solution:

Given: $y=(\cos x)^{x}+\tan^{-1}\sqrt{x}$

As we know, $\text{If } y=f\left(a\left(x\right),x\right) \text{ then } \frac{\mathrm{d}\,y}{\mathrm{d}\,x}=f'\left(a\left(x\right),x\right)\frac{\mathrm{d}\,a(x)}{\mathrm{d}\,x}$

Differentiating w.r.t. *x*, we get

$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=(\cos x)^{x}\ln\left(\cos x\right)\times\left(-\sin x\right)+\frac{1}{1+\left(\sqrt{x}\right)^{2}}\times\frac{1}{2\sqrt{x}}$ $\left[\because\frac{\mathrm{d}\,a^{x}}{\mathrm{d}\,x}=a^{x}\ln a,\ \frac{\mathrm{d}\left(\tan^{-1}x\right)}{\mathrm{d}\,x}=\frac{1}{1+x^{2}}\right]$

$\Rightarrow\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=-\sin x\left[\ln\left(\cos x\right)\right](\cos x)^{x}+\frac{1}{2\sqrt{x}(1+x)}$

Question 29

If $\overrightarrow{a}=\hat{i}+2\hat{j}+3\hat{k}\text{ and }\overrightarrow{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.

OR

Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, –1, 4) and C(4, 5, –1).

Solution:

Given, $\overrightarrow{a}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{b}=2\hat{i}+4\hat{j}-5\hat{k}$ represent two adjacent sides of a parallelogram.

Then, $\left(\overrightarrow{a}+\overrightarrow{b}\right)$ and $\left(\overrightarrow{a}-\overrightarrow{b}\right)$ represent diagonals of the parallelogram.

$\left(\overrightarrow{a}+\overrightarrow{b}\right)=\left(\hat{i}+2\hat{j}+3\hat{k}\right)+\left(2\hat{i}+4\hat{j}-5\hat{k}\right)=3\hat{i}+6\hat{j}-2\hat{k}$ and

$\left(\overrightarrow{a}-\overrightarrow{b}\right)=\left(\hat{i}+2\hat{j}+3\hat{k}\right)-\left(2\hat{i}+4\hat{j}-5\hat{k}\right)=-\hat{i}-2\hat{j}+8\hat{k}$

$\left|\overrightarrow{a}+\overrightarrow{b}\right|=\sqrt{3^{2}+6^{2}+(-2)^{2}}=7$ and $\left|\overrightarrow{a}-\overrightarrow{b}\right|=\sqrt{(-1)^{2}+(-2)^{2}+(8)^{2}}=\sqrt{69}$

Let $\overrightarrow{c}$ and $\overrightarrow{d}$ be the unit vectors parallel to the diagonals of the parallelogram respectively.

Then, $\overrightarrow{c}=\frac{\left(\overrightarrow{a}+\overrightarrow{b}\right)}{\left|\overrightarrow{a}+\overrightarrow{b}\right|}$ and $\overrightarrow{c}=\frac{\left(\overrightarrow{a}-\overrightarrow{b}\right)}{\left|\overrightarrow{a}-\overrightarrow{b}\right|}$

$\Rightarrow\overrightarrow{c}=\frac{3\hat{i}+6\hat{j}-2\hat{k}}{7}$ and $\overrightarrow{d}=\frac{-\hat{i}-2\hat{j}+8\hat{k}}{\sqrt{69}}$

OR

Given, $A(1,\ 2,\ 3),\ B(2,\ -1,4)$ and $C(4,\ 5,\ -1)$ are the vertices of triangle ABC.
Then, $\overrightarrow{AB}=(2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}=\hat{i}-3\hat{j}+\hat{k}$ and $\overrightarrow{AC}=(4-1)\hat{i}+(5-2)\hat{j}+(-1-3)\hat{k}=3\hat{i}+3\hat{j}-4\hat{k}$
Area of $\Delta ABC=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$$=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}$$
$$=\frac{1}{2}\left|(12-3)\hat{i}-(-4-3)\hat{j}+(3+9)\hat{k}\right|$$
$$=\frac{1}{2}\left|9\hat{i}+7\hat{j}+12\hat{k}\right|$$
$$=\frac{1}{2}\sqrt{9^2+7^2+12^2}$$
$$=\frac{1}{2}\sqrt{274}\text{ sq. units}$$

Question 30

A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A requires 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and and 8 minutes each for assembling. Given that total time for cutting is 3 hours 20 minutes and for assembling 4 hours. The profit for type A souvenir is ₹ 100 each and for type B souvenir, profit is ₹ 120 each. How many souvenirs of each type should the company manufacture in order to maximize the profit? Formulate the problem as an LPP and solve it graphically.

Solution:

Let the company manufacture *x* souvenirs of type *A* and *y* souvenirs of type *B*.
Number of items cannot be negative.

Therefore,
$x \geq 0$ and $y \geq 0$
The given information can be complied in a table as follows.

	Type *A*	Type *B*	Availability
Cutting(min)	5	8	3 × 60 + 20 = 200
Assembling(min)	10	8	4 × 60 = 240

Therefore, the constraints are

$$5x+8y\leq 200$$
$$10x+8y\leq 240$$

The profit on type *A* souvenirs is 100 rupees each and on type *B* souvenirs is 120 rupees each. Therefore, profit gained on *x* souvenirs of type *A* and *y* souvenirs of type *B* is Rs $100x$ and Rs $120y$ respectively.

Total profit, $Z = 100x + 120y$

The mathematical formulation of the given problem is
Maximize Z = 100x + 120y

subject to the constraints,

$5x+8y \le 200$
$10x+8y \le 240$
$x \ge 0$ and $y \ge 0$

First we will convert inequations into equations as follows:
5x + 8y = 200, 10x + 8y = 240, x = 0 and y = 0

Region represented by 5x + 8y ≤ 200:
The line 5x + 8y = 200 meets the coordinate axes at A_1(40, 0) and B_1(0, 25) respectively.

By joining these points we obtain the line 5x + 8y = 200. Clearly (0,0) satisfies the 5x + 8y = 200. So, the region which contains the origin represents the solution set of the inequation 5x + 8y ≤ 200.

Region represented by 10x + 8y ≤ 240:

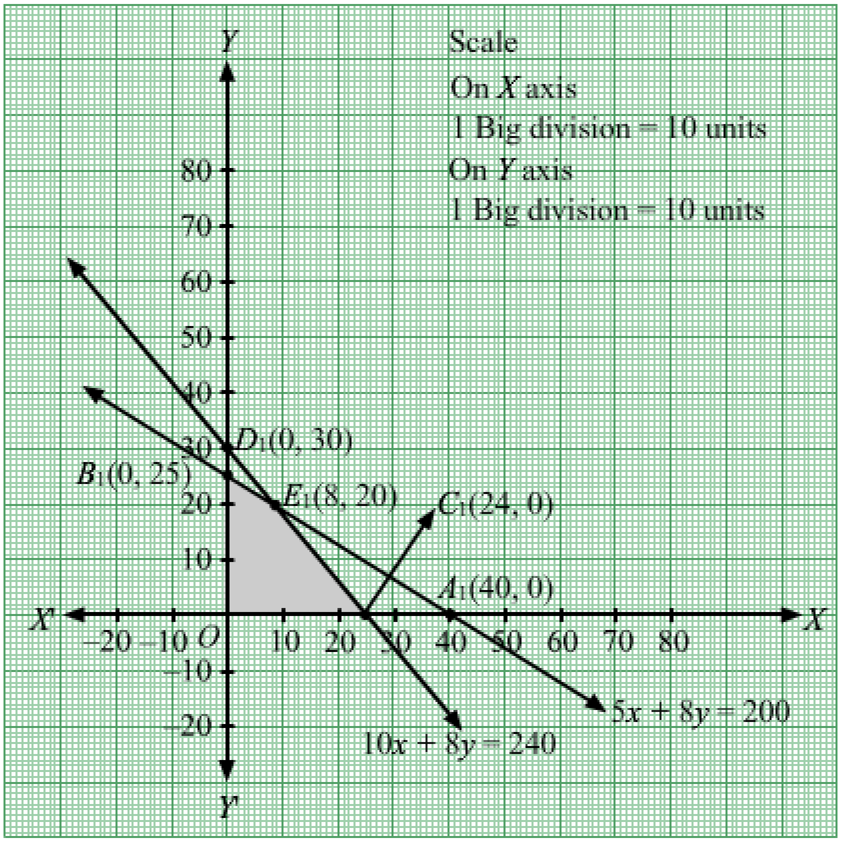
The line 10x + 8y = 240 meets the coordinate axes at C_1(24, 0) and D_1(0, 30) respectively. By joining these points we obtain the line

10x + 8y = 240. Clearly (0,0) satisfies the inequation 10x + 8y ≤ 240. So, the region which contains the origin represents the solution set of the inequation 10x + 8y ≤ 240.

Region represented by x ≥ 0 and y ≥ 0:

Since, every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the inequations x ≥ 0, and y ≥ 0.

The feasible region determined by the system of constraints 5x + 8y ≤ 200, 10x + 8y ≤ 240, x ≥ 0 and y ≥ 0 are as follows.



The corner points of the feasible region are $O(0, 0)$, $B_1(0, 25)$, $E_1(8, 20)$, $C_1(24, 0)$.
The values of Z at these corner points are as follows.

Corner point	$Z = 100x + 120y$
$O(0,0)$	0
$B_1(0,25)$	3000
$E_1(8,20)$	3200
$C_1(24, 0)$	2400

The maximum value of Z is 3200 at $E_1(8,20)$.

Thus, 8 souvenirs of type *A* and 20 souvenirs of type *B* should be produced each day to get the maximum profit of Rs 3200.

Question 31

Three rotten apples are mixed with seven fresh apples. Find the probability distribution of the number of rotten apples, if three apples are drawn one by one with replacement. Find the mean of the number of rotten apples.

OR

In a shop X, 30 tins of ghee of type A and 40 tins of ghee of type B which look alike, are kept for sale. While in shop Y, similar 50 tins of ghee of type A and 60 tins of ghee of type B are there. One tin of ghee is purchased from one of the randomly selected shop and is found to be of type B. Find the probability that it is purchased from shop Y.

Solution:

Let *X* represents number of rotten apples.

Therefore, $X = 0, 1, 2, 3$

$P(X = 0) = \frac{7}{10} \times \frac{7}{10} \times \frac{7}{10} = \frac{343}{1000}$

$P(X = 1) = {}^3C_1 \times \frac{3}{10} \times \frac{7}{10} \times \frac{7}{10} = 3 \times \frac{147}{1000} = \frac{441}{1000}$

$P(X = 2) = {}^3C_2 \times \frac{3}{10} \times \frac{3}{10} \times \frac{7}{10} = 3 \times \frac{63}{1000} = \frac{189}{1000}$

$P(X = 3) = {}^3C_3 \times \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10} = \frac{27}{1000}$

Hence, the required probability distribution is as follows:

X	0	1	2	3
$P(X)$	$\frac{343}{1000}$	$\frac{441}{1000}$	$\frac{189}{1000}$	$\frac{27}{1000}$

Mean of probability distribution = $0 \times \frac{343}{1000} + 1 \times \frac{441}{1000} + 2 \times \frac{189}{1000} + 3 \times \frac{27}{1000}$

$= \frac{441}{1000} + \frac{378}{1000} + \frac{81}{1000} = \frac{900}{1000} = \frac{9}{10}$

Hence, the required mean is $\frac{9}{10}$.

OR

Given:

	Shop *X*	Shop *Y*
Ghee of type A	30 tins	50 tins
Ghee of type B	40 tins	60 tins

We have $P(X) = \frac{1}{2}, P(Y) = \frac{1}{2}, P\left(B/X\right) = \frac{40}{70} = \frac{4}{7}$ and $P\left(B/Y\right) = \frac{60}{110} = \frac{6}{11}$

Therefore, $P\left(Y/B\right) = \frac{P(Y \cap B)}{P(B)}$

$$= \frac{P(Y) \cdot P\left(B/Y\right)}{P(Y) \cdot P\left(B/Y\right) + P(X) \cdot P\left(B/X\right)}$$

$$= \frac{\frac{1}{2} \times \frac{6}{11}}{\frac{1}{2} \times \frac{6}{11} + \frac{1}{2} \times \frac{4}{7}}$$

$$= \frac{\frac{3}{11}}{\frac{3}{11} + \frac{2}{7}}$$

$$= \frac{3}{11} \times \frac{77}{43} = \frac{21}{43}$$

Hence, the probability that the ghee was purchased from shop *Y* is $\frac{21}{43}$.

Question 32

Solve the differential equation:

$$x\ \sin\ \left(\frac{y}{x}\right)\frac{dy}{dx}+x-y\ \sin\ \left(\frac{y}{x}\right)=0$$

Given that x = 1 when $y=\frac{\pi}{2}$.

Solution:

The given differential equation is $x\sin\left(\frac{y}{x}\right)\frac{\mathrm{d}\,y}{\mathrm{d}\,x}+x-y\sin\left(\frac{y}{x}\right)=0$.

$\Rightarrow\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=\frac{y}{x}-\frac{1}{\sin\left(\frac{y}{x}\right)}\qquad\ldots\ldots(1)$

Put $\frac{y}{x}=v$ i.e. $y=vx$

$\Rightarrow\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=v+x\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$

From (1), we have

$\Rightarrow v+x\frac{\mathrm{d}\,v}{\mathrm{d}\,x}=v-\frac{1}{\sin\,v}$

$\Rightarrow x\frac{\mathrm{d}\,v}{\mathrm{d}\,x}=-\frac{1}{\sin\,v}$

$\Rightarrow\sin v\,\mathrm{d}\,v=-\frac{\mathrm{d}\,x}{x}$

$\Rightarrow\int\sin v\,\mathrm{d}\,v=-\int\frac{\mathrm{d}\,x}{x}$

$\Rightarrow-\cos v=-\log x-c$

$\Rightarrow\cos v=\log x+c$

$\Rightarrow\cos\left(\frac{y}{x}\right)=\log x+c\qquad\ldots\ldots(2)$

At x = 1 and $y=\frac{\pi}{2}$, (2) becomes

$\cos\left(\frac{\pi}{2}\right)=\log(1)+c$

$\Rightarrow 0=0+c$

$\Rightarrow c=0$

Thus, (2) reduces to $\cos\left(\frac{y}{x}\right)=\log x$.

Question 33

If *a, b, c* are p^{th}, q^{th} and r^{th} terms respectively of a G.P, then prove that $\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$

OR

If $\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1}.

Using A^{-1}, solve the following system of equations :

$2x - 3y + 5z = 11$

$3x + 2y - 4z = -5$

$x + y - 2z = -3$

Solution:

Let x and y are the first term and common ratio respectively.

Then

$a = a_p = xy^{p-1}$, ...(1)

$b = a_q = xy^{q-1}$ .....(2) and

$c = a_r = xy^{r-1}$ .....(3)

Dividing (2) by (1), we get

$\frac{b}{a} = \frac{y^{q-1}}{y^{p-1}} = y^{q-p} \Rightarrow \log\left(\frac{b}{a}\right) = \left(q-p\right)\log y$ .....(4)

Dividing (3) by (2), we get

$\frac{c}{b} = \frac{y^{r-1}}{y^{q-1}} = y^{r-q} \Rightarrow \log\left(\frac{c}{b}\right) = \left(r-q\right)\log y$ .....(5)

Dividing (1) bt (3), we get

$\frac{a}{c} = \frac{y^{p-1}}{y^{r-1}} = y^{p-r} \Rightarrow \log\left(\frac{a}{c}\right) = \left(p-r\right)\log y$ .....(6)

Now,

$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}$

Expanding along C_1,

$= \log a(q-r) - \log b(p-r) + \log c(p-q)$

$= q\log a - r\log a - p\log b + r\log b + p\log c - q\log c$

$= q(\log a - \log c) + p(\log c - \log b) + r(\log b - \log a)$

$$= q\log\left(\frac{a}{c}\right) + p\log\left(\frac{c}{b}\right) + r\log\left(\frac{b}{a}\right)$$
$$= q(p-r)\log y + p(r-q)\log y + r(q-p)\log y \quad \text{\{from (4), (5) and (6)\}}$$
$$= \log y\,(pq - qr + pr - pq + qr - pr)$$
$$= 0$$

Hence proved.

OR

Now, $A_{11} = 0,\ A_{12} = 2,\ A_{13} = 1$

$A_{21} = -1,\ A_{22} = -9,\ A_{23} = -5$

$A_{31} = 2,\ A_{32} = 23,\ A_{33} = 13$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad ...(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

The solution of the system of equations is given by $X = A^{-1}B$.

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad [\text{Using } (1)]$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1,\ y = 2,$ and $z = 3$.

Question 34

Find the vector and cartesian equations of the line which perpendicular to the lines with equations

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4} \text{ and } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and passes through the point (1, 1, 1). Also find the angle between the given lines.

Solution:

The given two equations are $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$.
The direction ratios of these lines are $(1,\ 2,\ 4)$ and $(2,\ 3,\ 4)$ respectively.

The direction ratio of the line which is perpendicular to both these lines is given by their cross product i.e. $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & 3 & 4 \end{vmatrix} = -4\hat{i}+4\hat{j}-\hat{k}$.

Now the equation of perpendicular passing through the point (1, 1, 1) and having direction ratios $-4\hat{i}+4\hat{j}-\hat{k}$ is given by
$\vec{r}=\left(\hat{i}+\hat{j}+\hat{k}\right)+\lambda\left(-4\hat{i}+4\hat{j}-\hat{k}\right)$

Its cartesian equation is given by $\frac{x-1}{-4}=\frac{y-1}{4}=\frac{z-1}{-1}$ or $\frac{1-x}{4}=\frac{y-1}{4}=\frac{1-z}{1}$.

Now, the angle between two given lines is given by
$\cos\theta=\left|\frac{1\times2+2\times3+4\times4}{\sqrt{1^2+2^2+4^2}\times\sqrt{2^2+3^2+4^2}}\right|=\frac{24}{\sqrt{21}\sqrt{29}}$

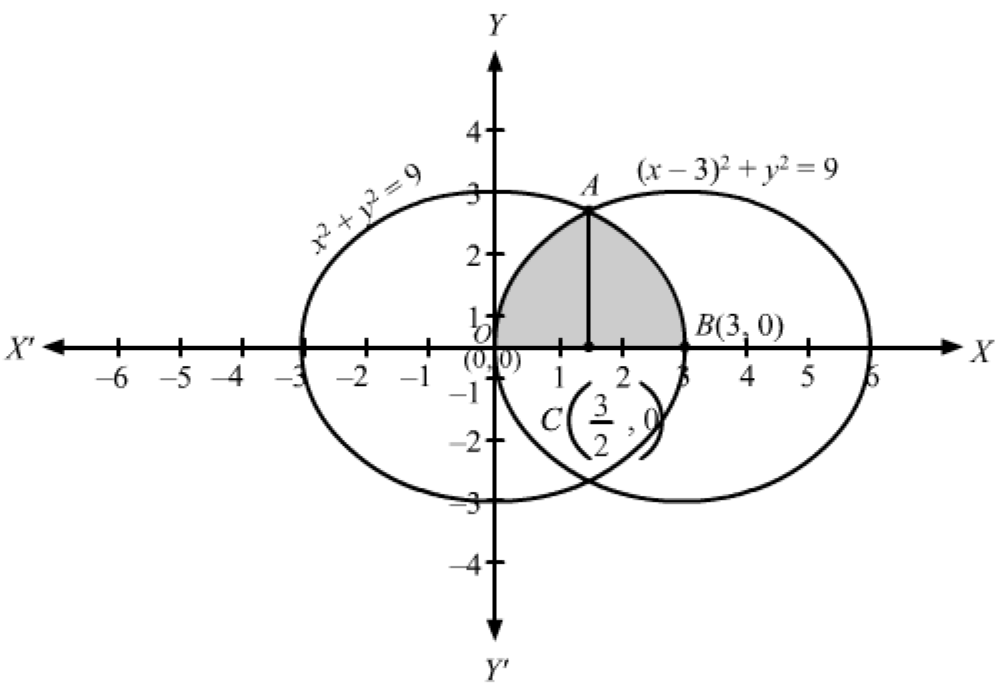
Question 35

Using integration find the area of the region bounded between the two circles $x^2+y^2=9$ and $(x-3)^2+y^2=9$.

OR

Evaluate the following integral as the limit of sums $\int_1^4\left(x^2-x\right)dx$.

Solution:



Let the two curves be named as y_1 and y_2 where

$y_1 : (x-3)^2 + y^2 = 9 \qquad .....(1)$

$y_2 : x^2 + y^2 = 9 \qquad .....(2)$

The curve $x^2 + y^2 = 9$ represents a circle with centre (0, 0) and the radius is 3.

The curve $(x - 3)^2 + y^2 = 9$ represents a circle with centre (3, 0) and has a radius 3.

To find the intersection points of two curves equate them.

On solving (1) and (2) we get

$x = \frac{3}{2} \text{ and } y = \pm\frac{3\sqrt{3}}{2}$

Therefore, intersection points are $\left(\frac{3}{2},\ \frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{3}{2},\ -\frac{3\sqrt{3}}{2}\right)$.

Now, the required area =2[area(*OACO*) +area(*CABC*)]

Here,

$\text{Area}\,(OACO) = \int_0^{\frac{3}{2}} y_1\, \mathrm{d}\,x$

$= \int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\, \mathrm{d}\,x$

And

$\text{Area}\,(CABC) = \int_{\frac{3}{2}}^{3} y_2.\, \mathrm{d}\,x$

$= \int_{\frac{3}{2}}^{3} \sqrt{9-x^2}dx$

Thus the required area is given by,

A = 2[area(*OACO*) +area(*CABC*)]

$= 2\left(\int_0^{\frac{3}{2}} \sqrt{9-(x-3)^2}\ \mathrm{d}\,x\ +\ \int_{\frac{3}{2}}^{3} \sqrt{9-x^2}\,\mathrm{d}\,x\right)$

$= 2\left[\frac{(x-3)}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right]_0^{\frac{3}{2}} + 2\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{2}}^{3}$

$= 2\left[\frac{\frac{3}{2}-3}{2}\sqrt{9-\left(\frac{3}{2}-3\right)^2} + \frac{9}{2}\sin^{-1}\left(\frac{\frac{3}{2}-3}{3}\right) - \frac{0-3}{2}\sqrt{9-(0-3)^2} - \frac{9}{2}\sin^{-1}\left(\frac{0-3}{3}\right)\right]$

$+ 2\left[\frac{3}{2}\sqrt{9-3^2} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right) - \left(\frac{3}{4}\sqrt{9-\frac{9}{4}}\right) - \frac{9}{2}\sin^{-1}\left(\frac{\frac{3}{2}}{3}\right)\right]$

$= 2\left[-\frac{9\sqrt{3}}{8} - \frac{9\pi}{12} + \frac{9\pi}{4}\right] + 2\left[\frac{9\pi}{4} - \frac{9\sqrt{3}}{8} - \frac{9\pi}{12}\right]$

$= -\frac{18\sqrt{3}}{8} - \frac{18\pi}{12} + \frac{18\pi}{4} + \frac{18\pi}{4} - \frac{18\sqrt{3}}{8} - \frac{18\pi}{12}$

$= \frac{-36\sqrt{3}}{8} - \frac{36\pi}{12} + \frac{36\pi}{4}$

$= -\frac{9\sqrt{3}}{2} - 3\pi + 9\pi$

$= 6\pi - \frac{9\sqrt{3}}{2}$

Hence the required area is $\left(6\pi - \frac{9\sqrt{3}}{2}\right)$ square units.

OR

Let $I=\int_1^4\left(x^2-x\right)dx$

$$=\int_1^4 x^2\,dx-\int_1^4 x\,dx$$

Let $I=I_1-I_2$, where $I_1=\int_1^4 x^2dx$ and $I_2=\int_1^4 x\,dx$ ...(1)

It is known that,

$\int_a^b f(x)\,\mathrm{d}x=(b-a)\lim_{n\to\infty}\frac{1}{n}\left[f(a)+f(a+h)+\ldots+f(a+(n-1)h)\right]$, where $h=\frac{b-a}{n}$

For $I_1=\int_1^4 x^2dx$,

$a=1, b=4,$ and $f(x)=x^2$

$\therefore h=\dfrac{4-1}{n}=\dfrac{3}{n}$

$$I_1=\int_1^4 x^2dx=(4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\ldots+f\left(1+(n-1)h\right)\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left(1+\frac{3}{n}\right)^2+\left(1+2\cdot\frac{3}{n}\right)^2+\ldots\left(1+\frac{(n-1)3}{n}\right)^2\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[1^2+\left\{1^2+\left(\frac{3}{n}\right)^2+2\cdot\frac{3}{n}\right\}+\ldots+\left\{1^2+\left(\frac{(n-1)3}{n}\right)^2+\frac{2\cdot(n-1)\cdot 3}{n}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[\left(\underbrace{1^2+\ldots+1^2}_{n\text{ times}}\right)+\left(\frac{3}{n}\right)^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}+2\cdot\frac{3}{n}\left\{1+2+\ldots+(n-1)\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9}{n^2}\left\{\frac{(n-1)(n)(2n-1)}{6}\right\}+\frac{6}{n}\left\{\frac{(n-1)(n)}{2}\right\}\right]$$

$$=3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{9n}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+\frac{6n-6}{2}\right]$$

$$=3\lim_{n\to\infty}\left[1+\frac{9}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)+3-\frac{3}{n}\right]$$

$$=3[1+3+3]$$

$$=3[7]$$

$I_1=21$ ...(2)

For $I_2=\int_1^4 x dx$,

$a=1, b=4,$ and $f(x)=x$

$\Rightarrow h=\dfrac{4-1}{n}=\dfrac{3}{n}$

$$\therefore I_2 = (4-1)\lim_{n\to\infty}\frac{1}{n}\left[f(1)+f(1+h)+\dots f\left(a+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+(1+h)+\dots+\left(1+(n-1)h\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[1+\left(1+\frac{3}{n}\right)+\dots+\left\{1+(n-1)\frac{3}{n}\right\}\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[\left(1+\underbrace{1+\dots}_{n\text{ times}}+1\right)+\frac{3}{n}\left(1+2+\dots+(n-1)\right)\right]$$

$$= 3\lim_{n\to\infty}\frac{1}{n}\left[n+\frac{3}{n}\left\{\frac{(n-1)n}{2}\right\}\right]$$

$$= 3\lim_{n\to\infty}\left[1+\frac{3}{2}\left\{1-\frac{1}{n}\right\}\right]$$

$$= 3\left[1+\frac{3}{2}\right]$$

$$= 3\left[\frac{5}{2}\right]$$

$$\Rightarrow I_2 = \frac{15}{2} \qquad \dots(3)$$

From equations (2) and (3), we obtain

$$I = I_1 + I_2 = 21 - \frac{15}{2} = \frac{27}{2}$$

Question 36

Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, 1).

Solution:

Given: $y^2 = 4x$
Let the abscissa of the point on the curve, which is nearest to the point (2, 1) be *t*.
$\text{Therefore, ordinate} = \sqrt{4t} = 2\sqrt{t}$

Let the distance between the points $\left(t,\ 2\sqrt{t}\right)$ and $(2,\ 1)$ be *d*.
Therefore,
$d = \sqrt{(t-2)^2 + \left(2\sqrt{t}-1\right)^2}$
$\Rightarrow d^2 = t^2 + 4 - 4t + 4t + 1 - 4\sqrt{t}$
$\Rightarrow d^2 = t^2 + 5 - 4\sqrt{t}$

Differentiating w.r.t. *t*, we get
$2d\frac{\mathrm{d}(d)}{\mathrm{d}\,t} = 2t + 0 - \frac{4}{2\sqrt{t}}$

$\Rightarrow \frac{\mathrm{d}(d)}{\mathrm{d}\,t} = \frac{2t - \frac{4}{2\sqrt{t}}}{2d}$
$\text{For minima, } 2t - \frac{4}{2\sqrt{t}} = 0$

$\Rightarrow 2t = \frac{4}{2\sqrt{t}}$

$\Rightarrow t = 1$
Hence, the point, which is nearest to the point $(2,\ 1)$ is $\left(1,\ 2\sqrt{1}\right) \text{ i.e. } (1,\ 2)$.