Mathematics 2019
Delhi Set-3

General Instructions:

(i) **All** questions are compulsory.

(ii) This question paper contains **29** questions divided into four sections A, B, C and D. Section **A** comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only **one** of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask logarithmic tables, if required.

Question 1

If $3\mathrm{A} - \mathrm{B} = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $\mathrm{B} = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A.

SOLUTION:

$3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$

We need to calculate A.

$3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + B$

$B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$

$3A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$

$A = \frac{1}{3}\begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$

$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

Question 2

Write the order and the degree of the following differential equation:

$$x^3\left(\frac{d^2y}{dx^2}\right)^2+x\left(\frac{dy}{dx}\right)^4=0$$

SOLUTION:

Order is the highest order derivative present in the differential equation
And degree is the power of highest order derivative.
We have given the differential equation:

$$x^3\left(\frac{d^2y}{d\,x^2}\right)^2+x\left(\frac{d\,y}{d\,x}\right)^4=0$$

Here, order is 2 and degree is 2.

Question 3

If $f(x) = x + 1$, find $\frac{d}{dx}(fof)\ (x)$.

SOLUTION:

Given: $f(x)=x+1$

$fof(x)=(x+1)+1=x+2$

$\frac{d}{dx}(fof)(x)=\frac{d}{dx}(x+2)=1$

Question 4

If a line makes angles 90°, 135°, 45° with the *x, y* and *z* axes respectively, find its direction cosines.

OR

Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{\imath}+2\hat{\jmath}-3\hat{k}$.

SOLUTION:

A line makes $90^\circ,\ 135^\circ,\ 45^\circ$ with $x,\ y$ and z axes respectively.
Therefore, Direction cosines of the line are $\cos 90^\circ,\ \cos 135^\circ\ \text{ and }\ \cos 45^\circ$
⇒ Direction cosines of the line are $0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}$

OR

Vector equation of a line which passes through a point $(3,4,5)$ and parallel to the vector $2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\widehat{\mathbf{k}}$ is

$$\overrightarrow{r}=3\hat{\mathbf{i}}+4\hat{\mathbf{j}}+5\widehat{\mathbf{k}}+\mu\left(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\widehat{\mathbf{k}}\right)$$

Question 5

Find : ∫ sin x · log cos x dx

SOLUTION:

$\int \sin x . \log \; \cos x \; dx$

Substitute cos$x$ = t

-sin$x$dx = dt

$$\int -\log t \; dt$$
$$= -(t \; \log t - t) + C$$
$$= -t \; \log t + t + C$$
$$= -\cos x \, log\,(cos\,x) + cos\,x + C$$

SOLUTION:

$\int_{-\pi}^{\pi} \left(1 - x^2\right) \sin x \cos^2 x \,\mathrm{d}\, x$

We know

$\int_{-a}^{a} f(x) \,\mathrm{d}\, x = 0$ if f is an odd function i.e if $f(-x) = -f(x)$

In the given integral,

$f(x) = \left(1 - x^2\right) \sin x \cos^2 x$

$\Rightarrow f(-x) = \left(1 - (-x)^2\right) (\sin(-x)) \cos^2(-x) = -\left(1 - x^2\right) \sin x \cos^2 x$

$\Rightarrow f(-x) = -f(x)$

So, $\int_{-\pi}^{\pi} \left(1 - x^2\right) \sin x \cos^2 x \,\mathrm{d}\, x$ = 0

OR

$|x|$= x when $x \geq 0$

= $-x$ when $x \; < \; 0$

Therefore, $\frac{|x|}{x}$ = 1 when x ≥ 0

= -1 when $x \; < \; 0$

Thus, $\int_{-1}^{2} \frac{|x|}{x} dx \; = \; \int_{-1}^{0} \left(-1\right) dx \; + \; \int_{0}^{2} \left(1\right) dx$

$= -1 \times [x]_{-1}^{0} + [x]_{0}^{2}$

$= (-1)\,[0 + 1] + [2 - 0] \; = \; -1 \; + \; 2 \; = \; 1.$ Answer

Question 7

Examine whether the operation *defined on R by $a * b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not ?

SOLUTION:

The given operation is $a * b = ab + 1$

If any operation is a binary operation then it must follow the closure property.

Let $a \in R$, $b \in R$

then $a*b \in R$

also $ab + 1 \in R$

i.e. $a * b \in R$

so * on R satisfies the closure property

Now if this binary operation satisfies associative law then

$(a * b) * c = a * (b * c)$

$$\begin{aligned}(a * b) * c &= (ab + 1) * c \\ &= (ab + 1)c + 1 \\ &= abc + c + 1\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a * (bc + 1) \\ &= a(bc + 1) + 1 \\ &= abc + a + 1\end{aligned}$$

$\therefore (a * b) * c \neq a * (b * c)$

i.e., * operation does not follow associative law.

Question 8

Find a matrix A such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

SOLUTION:

Given: $2A - 3B + 5C = 0$

$$\Rightarrow 2A - 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} + 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} 0$$

$$\Rightarrow 2A - \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix} = 0$$

$$\Rightarrow 2A + \begin{bmatrix} 10+6 & 0-6 & -10-0 \\ 35-9 & 5-3 & 30-12 \end{bmatrix} = 0$$

$$\Rightarrow 2A = -\begin{bmatrix} 16 & -6 & -10 \\ 26 & 2 & 18 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$$

SOLUTION:

Solution not Available

Question 10

Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where 'a' and 'b' are arbitrary constants.

SOLUTION:

Given: $y = e^{2x}(a + bx)$

Differentiating the above equation, we get

$$\frac{dy}{dx} = be^{2x} + 2(a + bx)e^{2x}$$

$$= \frac{dy}{dx} = be^{2x} + 2y \quad \ldots\text{(i)} \quad [\because y = e^{2x}(a + bx)]$$

differentiating the above equation, we get

$$\frac{d^2y}{dx^2} = 2\,be^{2x} + 2\frac{dy}{dx}$$

$$= \frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx} - 2y\right) + 2\frac{dy}{dx} \quad \left[\because \text{ from } (\text{i}) \text{ we get, } be^{2x} = \frac{dy}{dx} - 2y\right]$$

$$= \frac{d^2y}{dx^2} = 4\frac{dy}{dx} - 4y$$

Hence, the required differential equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$

Question 11

A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) atmost 5 successes?

OR

The random variable X has a probability distribution P(X) of the following form, where '*k*' is some number.

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of '*k*'.

SOLUTION:

Let's consider success as '*p*' and failure as '*q*'.

Here we have 3 odd and 3 even numbers out of total 6 sample spaces

P(*p*) = $\frac{1}{2}$ and P(*q*) = $\frac{1}{2}$

(i) Probability of getting 5 successes = P(*p*=5) = ${}^6C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^1 = {}^6C_5\left(\frac{1}{2}\right)^6 = \frac{3}{32}$.

(ii) Probability of getting atmost 5 successes = P($p \leq 5$) = 1 − P(*p*=6) = 1 − ${}^6C_6\left(\frac{1}{2}\right)^6$ = 1 − $\frac{1}{64}$ = $\frac{63}{64}$.

OR

It is known that the sum of probabilities of a probability distribution of random variables is one.

$\therefore k + 2k + 3k + 0 = 1$

$\Rightarrow 6k = 1$

$\Rightarrow k = \frac{1}{6}$

Question 12

If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

OR

If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k},\ \vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $\left[\vec{a}\ \vec{b}\ \vec{c}\right]$.

SOLUTION:

Let three unit vectors are *a, b* and *c*
given that the sum of the unit vectors is a unit vector.

$\therefore\ a+b=c$

or $|c|^2=|a+b|^2$

or $|c|^2=|a|^2+|b|^2+2|a||b|\cos\theta$

or $1 = 1 + 1 + 2\cos\theta$ [$\because |a| = |b| = |c| = 1$ (unit vector)]

$\Rightarrow \cos\theta = -1/2$ ...(1)

Now, $|a - b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$
$|a - b|^2 = [1+1+1]$
$|a - b| = \sqrt{3}$

OR

$$\left[\vec{a}\ \vec{b}\ \vec{c}\right]=\begin{vmatrix}2 & 3 & 1\\ 1 & -2 & 1\\ -3 & 1 & 2\end{vmatrix}=2(-4-1)-3(2+3)+1(1-6)$$
$$=-30$$

Question 13

Using properties of determinants, prove the following:

$$\begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ b+c & c+a & a+b\end{vmatrix}=a^3+b^3+c^3-3abc.$$

SOLUTION:

$$\Delta=\begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ b+c & c+a & a+b\end{vmatrix}$$

$$=\begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ a+b+c & c+a+b & a+b+c\end{vmatrix}\qquad [\text{Applying } R_3\to R_3+R_1]$$

$$=\left(a+b+c\right)\begin{vmatrix}a & b & c\\ a-b & b-c & c-a\\ 1 & 1 & 1\end{vmatrix}\qquad [\text{Taking } (\mathrm{a+b+c}) \text{ common}]$$

$$= \left(a+b+c\right)\begin{vmatrix} a & b & c \\ -b & -c & -a \\ 1 & 1 & 1 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1]$$

$$= \left(a+b+c\right)\begin{vmatrix} a-c & b-c & c \\ -b+a & -c+a & -a \\ 0 & 0 & 1 \end{vmatrix} \qquad [C_1 \to C_1 - C_3 \text{ and } C_2 \to C_2 - C_3]$$

$$= \left(a+b+c\right)[\{(a-c)(a-c)-(b-c)(a-b)\}]$$

$$= \left(a+b+c\right)\left[(a-c)^2-\left(ab-ac-b^2+bc\right)\right]$$

$$= \left(a+b+c\right)\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$$= a^3+b^3+c^3-3abc$$

Question 14

Solve: $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$.

SOLUTION:

We have $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$

$\Rightarrow \quad \tan\left(\tan^{-1} 4x + \tan^{-1} 6x\right) = \tan\frac{\pi}{4}$

$\Rightarrow \quad \frac{\tan\left(\tan^{-1} 4x\right)+\tan\left(\tan^{-1} 6x\right)}{1-\tan\left(\tan^{-1} 4x\right)\cdot\tan\left(\tan^{-1} 6x\right)} = 1$

$\Rightarrow \quad \frac{4x+6x}{1-4x\cdot 6x} = 1$

$\Rightarrow \quad \frac{10x}{1-24x^2} = 1$

$\Rightarrow \quad 24x^2+10x-1=0$

$\Rightarrow \quad 24x^2+12x-2x-1=0$

$\Rightarrow \quad 12x(2x+1)-1(2x+1)=0$

$\Rightarrow \quad (2x+1)(12x-1)=0$

$\Rightarrow \quad x=-\frac{1}{2}, \frac{1}{12}$

But $x=\frac{-1}{2}$ does not satisfy the equation as the LHS will become negative

Therefore, the value of x is $\frac{1}{12}$.

Question 15

Show that the relation R on $\mathbb{R}$ defined as R = {(*a*, *b*) : $a \le b$}, is reflexive, and transitive but not symmetric.

OR

Prove that the function $f: N \to N$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of $f: N \to S$, where S is range of *f*.

SOLUTION:

R = {(a, b); a ≤ b}

Clearly (a, a) ∈ R as a = a.

∴ R is reflexive.

Now,

(2, 4) ∈ R (as 2 < 4)

But, (4, 2) ∉ R as 4 is greater than 2.

∴ R is not symmetric.

Now, let (a, b), (b, c) ∈ R.

Then,

a ≤ b and b ≤ c

⇒ a ≤ c

⇒ (a, c) ∈ R

∴ R is transitive.

Hence, R is reflexive and transitive but not symmetric.

OR

The given function is

$f: N \to N$

$f(x) = x^2 + x + 1$

Let x_1, x_2 6N

$$\text{So let } f(x_1) = f(x_2)$$

$$x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$$

$$x_1^2 - x_2^2 + x_1 - x_2 = 0$$

$$(x_1 - x_2)(x_1 + x_2 + 1) = 0$$

$$\because x_2 = x_1$$

$$\text{or } x_2 = -x_1 - 1$$

$$\because x_1 \in N$$

$$\therefore -x_1 - 1 \in N$$

$$\text{So } x_2 \neq -x_1 - 1$$

$$\because f(x_2) = f(x_1) \text{ only for } x_1 = x_2$$

So $f(x)$ is one-one function.

$\because f(x) = x^2 + x + 1$

$f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

Which is an increasing function.

$f(1) = 3$

$\because$ Range of $f(x)$ will be {3, 7, ...}

Which is a subset of N.

So it is an into function.

i.e., $f(x)$ is not an onto function.

let $y = x^2 + x + 1$

$x^2 + x + 1 - y = 0$

$x = \frac{-1 \pm \sqrt{1-4(1-y)}}{2}$

$x = \frac{-1 \pm \sqrt{4y-3}}{2}$

So two possibilities are their for $f^{-1}(x)$

$f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}, \frac{-1-\sqrt{4x-3}}{2}$ and we know $f^{-1}(3) = 1$ because $f(1) = 3$

so $f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}$

Question 16

Find the equation of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact.

SOLUTION:

Slope of the given line is 2

Let (x_1, y_1) be the point where the tangent is drawn to the curve $y = \sqrt{3x-2}$

Since, the point lies on the curve.

Hence, $y_1 = \sqrt{3x_1 - 2} \quad \ldots (1)$

Now, $y = \sqrt{3x-2}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$

Slope of tangent at $(x_1, y_1) = \frac{3}{2\sqrt{3x_1-2}}$

Given that

Slope of tangent $=$ slope of the given line

$\Rightarrow \frac{3}{2\sqrt{3x_1-2}} = 2$

$\Rightarrow 3 = 4\sqrt{3x_1 - 2}$

$\Rightarrow 9 = 16\left(3x_1 - 2\right)$

$\Rightarrow \frac{9}{16} = 3x_1 - 2$

$\Rightarrow 3x_1 = \frac{9}{16} + 2 = \frac{9+32}{16} = \frac{41}{16}$

$\Rightarrow x_1 = \frac{41}{48}$

Now, $y_1 = \sqrt{\frac{123}{48} - 2} = \sqrt{\frac{27}{48}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$ [From (1)]

$\therefore (x_1,\ y_1) = \left(\frac{41}{48},\ \frac{3}{4}\right)$

Equation of tangent is,

$y - y_1 = m\ (x - x_1)$

$\Rightarrow y - \frac{3}{4} = 2\ \left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{4} = 2\left(\frac{48x-41}{48}\right)$

$\Rightarrow 24y - 18 = 48x - 41$

$\Rightarrow 48x - 24y - 23 = 0$

Equation of normal at the point of contact will be

$y - y_1 = \frac{-1}{m}\left(x - x_1\right)$

$\Rightarrow y - \frac{3}{4} = \frac{-1}{2}\left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{4} = \frac{-1}{2}\left(x - \frac{41}{48}\right)$

$\Rightarrow \frac{4y-3}{2} = \left(\frac{41}{48} - x\right)$

$\Rightarrow \frac{4y-3}{2} = \frac{41-48x}{48}$

$\Rightarrow 4y - 3 = \frac{41-48x}{24}$

$96y - 72 = 41 - 48x$

$\Rightarrow 48x + 96y = 113$

Question 17

If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

OR

If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

SOLUTION:

$$\text{We have, } \log\left(x^2+y^2\right) = 2\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{2}\log\left(x^2+y^2\right) = \tan^{-1}\left(\frac{y}{x}\right)$$

Differentiate with respect to x, we get,

$$\Rightarrow \frac{1}{2}\frac{d}{dx}\log\left(x^2+y^2\right) = \frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\frac{d}{dx}\left(x^2+y^2\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\left[2x+2y\frac{dy}{dx}\right] = \frac{x^2}{(x^2+y^2)}\left[\frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}\right]$$

$$\Rightarrow \left(\frac{1}{x^2+y^2}\right)\left(x+y\frac{dy}{dx}\right) = \frac{x^2}{(x^2+y^2)}\left[\frac{x\frac{dy}{dx}-y\frac{d}{dx}(x)}{x^2}\right]$$

$$\Rightarrow \left(\frac{1}{x^2+y^2}\right)\left(x+y\frac{dy}{dx}\right) = \frac{x^2}{(x^2+y^2)}\left[\frac{x\frac{dy}{dx}-y(1)}{x^2}\right]$$

$$\Rightarrow x+y\frac{dy}{dx} = x\frac{dy}{dx}-y$$

$$\Rightarrow y\frac{dy}{dx}-x\frac{dy}{dx} = -y-x$$

$$\Rightarrow \frac{dy}{dx}(y-x) = -(y+x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(y+x)}{y-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

OR

The given function is $x^y - y^x = a^b$

Let $x^y = u$ and $y^x = v$

Then, the function becomes $u - v = a^b$

$$\frac{du}{dx} - \frac{dv}{dx} = 0 \qquad \ldots\ldots(1)$$

$$u = x^y$$
$$\Rightarrow \log u = \log\left(x^y\right)$$
$$\Rightarrow \log u = y \log x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x \frac{dy}{dx} + y \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x \frac{dy}{dx} + y \cdot \frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) \qquad \ldots(2)$$

$$v = y^x$$
$$\Rightarrow \log v = \log\left(y^x\right)$$
$$\Rightarrow \log v = x \log y$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(\log y)$$
$$\Rightarrow \frac{dv}{dx} = v\left(\log y \cdot 1 + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right)$$
$$\Rightarrow \frac{dv}{dx} = y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$x^y\left(\log x \frac{dy}{dx} + \frac{y}{x}\right) - y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) = 0$$
$$\Rightarrow x^y \log x \frac{dy}{dx} - xy^{x-1}\frac{dy}{dx} + x^{y-1}y - y^x \log y = 0$$
$$\Rightarrow \left(x^y \log x - xy^{x-1}\right)\frac{dy}{dx} = y^x \log y - x^{y-1}y$$
$$\Rightarrow \frac{dy}{dx} = \frac{y^x \log y - x^{y-1}y}{\left(x^y \log x - xy^{x-1}\right)}$$

Question 18

If $y = (\sin^{-1}x)^2$, prove that $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$.

SOLUTION:

Here,

$$y = \left(\sin^{-1}x\right)^2$$

Now,

$$y_1 = 2\ \sin^{-1}x\ \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_2 = \frac{2}{1-x^2} + \frac{2x\ \sin^{-1}x}{\left(1-x^2\right)^{3/2}}$$

$$\Rightarrow y_2 = \frac{2}{1-x^2} + \frac{2x\ \sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}$$

$$\Rightarrow y_2 = \frac{2}{1-x^2} + \frac{xy_1}{(1-x^2)}$$

$$\Rightarrow y_2\left(1-x^2\right) = 2 + xy_1$$

$$\Rightarrow y_2\left(1-x^2\right) - xy_1 - 2 = 0$$

Therefore, $\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$

Hence proved.

Question 19

Prove that $\int_0^a f(x)\,\mathrm{d}\,x = \int_0^a f(a-x)\,\mathrm{d}\,x$, hence evaluate $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}\,x$.

SOLUTION:

To prove: $\int_0^a f(x)\,\mathrm{d}x = \int_0^a f(a-x)\,\mathrm{d}x$

Proof: Let

$t = a - x$

$\Rightarrow dt = -dx$

When $x = 0$, $t = a$

When $x = a$, $t = 0$

Putting the value of x in LHS

$$\int_a^0 f(a-t)\,(-\mathrm{d}t)$$

$$= -\int_a^0 f(a-t)\,(\mathrm{d}t)$$

$$= \int_0^a f(a-t)\,(\mathrm{d}t)$$

$$= \int_0^a f(a-x)\,(\mathrm{d}x) \qquad \left(\because \int_a^b f(t)\,\mathrm{d}t = \int_a^b f(x)\,\mathrm{d}x\right)$$

$$= \mathrm{RHS}$$

Using this we can solve the given question as follows:

$$I = \int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$$

$$\Rightarrow 2I = \int_0^{\pi} f(x)dx + \int_0^{\pi} f(\pi - x)\,dx = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2(\pi - x)}dx$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$$

$$Let, \cos x = t \Rightarrow -\sin x\,dx = dt$$

$$\Rightarrow 2I = -\int_1^{-1} \frac{\pi}{1+t^2}dt = -\pi\left[\tan^{-1} t\right]_1^{-1} = -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$$

$$\therefore I = \frac{\pi^2}{4}$$

Question 20

Find: $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$

SOLUTION:

$\int \frac{\cos x}{(1+\sin x)(2+\sin x)} dx$

Put 2 + sin*x* = *t*

$\Rightarrow 1+\sin x = t-1$

cos*xdx* = *dt*

$\int \frac{dt}{(t-1)t}$

$= \int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$

$= \int \frac{1}{t-1} dt - \int \frac{1}{t} dt$

$= \log(t-1) - \log t + C$

$= \log(2+\sin x - 1) - \log(2+\sin x) + C$

$= \log(1+\sin x) - \log(2+\sin x) + C$

$= \log\left(\frac{1+\sin x}{2+\sin x}\right) + C \quad \left(\because \log m - \log n = \log\left(\frac{m}{n}\right)\right)$

Question 21

Solve the differential equation: $\frac{dy}{dx} - \frac{2x}{1+x^2} y = x^2 + 2$

OR

Solve the differential equation: $(x+1)\frac{dy}{dx} = 2e^{-y} - 1; y(0) = 0.$

SOLUTION:

The given differential equation is

$$\frac{dy}{dx} - \frac{2x}{(1+x^2)} y = (x^2+2)$$

This equation is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-2x}{1+x^2}$ and $Q = x^2 + 2$

$$\text{Now, I.F} = e^{\int P dx} = e^{\int \frac{-2x}{1+x^2} dx} = e^{-\log(1+x^2)} = e^{\log\left(\frac{1}{1+x^2}\right)} = \frac{1}{1+x^2}$$

The general solution of the given differential equation is

$y \times \text{I.F.} = \int (\text{Q} \times \text{I.F.})\, dx + \text{C}$, where C is an aribatry constant

$$\Rightarrow \frac{y}{1+x^2} = \int \frac{x^2+2}{1+x^2} dx + \text{C}$$
$$= \int \left(1 + \frac{1}{x^2+1}\right) dx + \text{C}$$
$$= \int dx + \int \frac{1}{x^2+1} dx + \text{C}$$
$$= x + \tan^{-1} x + \text{C}$$
$$y = (1+x^2)(x + \tan^{-1} x + \text{C})$$

OR

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$
$$\Rightarrow \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1}$$
$$\Rightarrow \frac{e^y dy}{2-e^y} = \frac{dx}{x+1}$$

Integrating both sides, we get:

$$\int \frac{e^y dy}{2-e^y} = \log|x+1| + \log \text{C} \qquad ...(1)$$

Let $2 - e^y = t$.

$$\therefore \frac{d}{dy}\left(2-e^y\right) = \frac{dt}{dy}$$
$$\Rightarrow -e^y = \frac{dt}{dy}$$
$$\Rightarrow e^y dt = -dt$$

Substituting this value in equation (1), we get:

$$\int \frac{-dt}{t} = \log|x+1| + \log \text{C}$$
$$\Rightarrow -\log|t| = \log|\text{C}(x+1)|$$
$$\Rightarrow -\log|2-e^y| = \log|\text{C}(x+1)|$$
$$\Rightarrow \frac{1}{2-e^y} = \text{C}(x+1)$$
$$\Rightarrow 2 - e^y = \frac{1}{\text{C}(x+1)} \qquad ...(2)$$

Now, at $x = 0$ and $y = 0$, equation (2) becomes:

$$\Rightarrow 2-1=\frac{1}{C}$$
$$\Rightarrow C=1$$

Substituting C = 1 in equation (2), we get:

$$2-e^y=\frac{1}{x+1}$$
$$\Rightarrow e^y=2-\frac{1}{x+1}$$
$$\Rightarrow e^y=\frac{2x+2-1}{x+1}$$
$$\Rightarrow e^y=\frac{2x+1}{x+1}$$
$$\Rightarrow y=\log\left|\frac{2x+1}{x+1}\right|, (x\neq -1)$$

This is the required particular solution of the given differential equation.

Question 22

If $\hat{i}+\hat{j}+\hat{k}, 2\hat{i}+5\hat{j}, 3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively are the position vectors A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{CD}}$ are collinear or not.

SOLUTION:

Given:

The position vector of A is $\hat{\boldsymbol{i}}+\hat{\boldsymbol{j}}+\widehat{\boldsymbol{k}}$.

The position vector of B is $2\hat{\boldsymbol{i}}+5\hat{j}$.

Therefore, $\overrightarrow{AB}=(2-1)\hat{\boldsymbol{i}}+(5-1)\hat{j}+(0-1)\hat{k}=\hat{i}+4\hat{j}-\widehat{\boldsymbol{k}}$

The position vector of C is $3\hat{\boldsymbol{i}}+2\hat{j}-3\hat{k}$ and

The position vector of D is $\hat{\boldsymbol{i}}-6\hat{j}-\widehat{\boldsymbol{k}}$.

Therefore, $\overrightarrow{CD}=(1-3)\hat{i}+(-6-2)\hat{j}+(-1+3)\widehat{\boldsymbol{k}}=-2\hat{\boldsymbol{i}}-8\hat{j}+2\widehat{\boldsymbol{k}}$

Let θ be the angle between AB and CD, then

$$\cos\theta=\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}$$

$$\Rightarrow \cos\theta=\frac{-2-32-2}{\sqrt{18}\sqrt{72}}=-1$$

$$\Rightarrow \theta=180^\circ$$

since angle between Line AB and CD is 180°, therefore $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

Question 23

Find the value of λ, so that the lines $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not.

SOLUTION:

Given lines are $\frac{1-x}{3}=\frac{7y-14}{\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$

Converting them into standard form, we have $\frac{x-1}{-3}=\frac{y-2}{\left(\lambda/_{7}\right)}=\frac{z-3}{2}$ and $\frac{x-1}{\left(-3\lambda/_{7}\right)}=\frac{y-5}{1}=\frac{z-6}{-5}$

Corresponding d.r.'s are (($-3,\ \frac{\lambda}{7},\ 2$) and $\left(\frac{-3\lambda}{7},\ 1,\ -5\right)$

Since the angle between the lines is right angle so, $\cos 90^0=\left|\frac{(-3)\left(\frac{-3\lambda}{7}\right)+\left(\frac{\lambda}{7}\right)(1)+(2)(-5)}{\sqrt{(-3)^2+\left(\frac{\lambda}{7}\right)^2+2^2}\sqrt{\left(\frac{-3\lambda}{7}\right)^2+1^2+(-5)^2}}\right|$

$\Rightarrow 0=\left|\frac{\frac{9\lambda}{7}+\frac{\lambda}{7}-10}{\sqrt{\frac{\lambda^2}{49}+13}\sqrt{\frac{9\lambda^2}{49}+26}}\right|$

Squaring and cross-multiplying

$\Rightarrow \left(\frac{10\lambda}{7}-10\right)^2=0$

$\Rightarrow \frac{10\lambda}{7}=10$

$\Rightarrow \lambda=7.$

Substituting the value of λ, the lines are $\frac{x-1}{-3}=\frac{y-2}{1}=\frac{z-3}{2}=a\left(\text{let}\right)$ and $\frac{x-1}{-3}=\frac{y-5}{1}=\frac{z-6}{-5}=b$(let)

From first equation, $(x,\ y,\ z)\ =\ (-3a+1,\ a+2,\ 2a+3)$ and from second equation,

$(x,\ y,\ z)\ =\ (-3b+1,\ b+5,\ -5b+6)$

Equating the corresponding values of coordinates, we have

$-3a+1\ =\ -3b\ +\ 1,\ a+2\ =\ b+5\ \text{and}\ 2a+3\ =\ -5b+6$

Or, $-3a\ +\ 3b\ =\ 0,\ a\ -\ b\ =3\ \text{and}\ 2a\ +\ 5b\ =\ 3$

Solving second and third equations of the above, we get $a=\frac{18}{7}$ and $b\ =\frac{-3}{7}$

Substituting these values of a and b in the first one

$-3\left(\frac{18}{7}\right)+3\left(\frac{-3}{7}\right)=-9$

Thus, it is clear that the first equation is not satisfied so the lines are not intersecting.

Question 24

A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank?

SOLUTION:

Let l, b and h be the length, breadth and height of the tank, respectively.

Height, h = 2 m

Volume of the tank = 8 m^3

Volume of the tank = $l \times b \times h$

$\therefore\ l \times b \times 2 = 8$

$$\Rightarrow lb = 4$$
$$\Rightarrow b = \frac{4}{l}$$

Area of the base = lb = 4 m^2

Area of the 4 walls, $A = 2h\,(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$
$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

For maximum or minimum values of A, we must have

$$\frac{dA}{dl} = 0$$
$$\Rightarrow 4\left(1 - \frac{4}{l^2}\right) = 0$$
$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Thus,
l = 2 m

$$\therefore\ b = \frac{4}{2} = 2\ \mathrm{m}$$

Now,

$$\frac{d^2A}{dl^2} = \frac{32}{l^3}$$

At $l = 2$:

$$\frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0$$

Thus, the area is the minimum when l = 2 m

We have
$l = b = h = 2$ m

∴ Cost of building the base = Rs 70 × (lb) = Rs 70 × 4 = Rs 280

Cost of building the walls = Rs $2h$ (l + b) × 45 = Rs 90 (2) (2 + 2)= Rs 8 (90) = Rs 720

Total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Question 25

If $\mathrm{A}=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}$, find A^{-1}. Hence, solve the system of equations $x + y + z = 6$, $x + 2z = 7$, $3x + y + z = 12$.

OR

Find the inverse of the following matrix using elementary operations.

$$\mathrm{A}=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$$

SOLUTION:

Given Matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 3 & 1 & 1\end{bmatrix}$

To find A^{-1}, we need cofactors of each element of matrix A.

$$\text{cofactor of } a_{11}=(-1)^{1+1}\begin{vmatrix}0 & 2\\ 1 & 1\end{vmatrix}=-2$$

$$\text{cofactor of } a_{12}=(-1)^{1+2}\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix}=-(1-6)=5$$

$$\text{cofactor of } a_{13}=(-1)^{1+3}\begin{vmatrix}1 & 0\\ 3 & 1\end{vmatrix}=1$$

$$\text{cofactor of } a_{21} = (-1)^{2+1}\begin{vmatrix}1 & 1\\1 & 1\end{vmatrix} = 0$$

$$\text{cofactor of } a_{22} = (-1)^{2+2}\begin{vmatrix}1 & 1\\3 & 1\end{vmatrix} = (1-3) = -2$$

$$\text{cofactor of } a_{23} = (-1)^{2+3}\begin{vmatrix}1 & 1\\3 & 1\end{vmatrix} = -(1-3) = 2$$

$$\text{cofactor of } a_{31} = (-1)^{3+1}\begin{vmatrix}1 & 1\\0 & 2\end{vmatrix} = 2$$

$$\text{cofactor of } a_{32} = (-1)^{3+2}\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix} = -(2-1) = -1$$

$$\text{cofactor of } a_{33} = (-1)^{3+3}\begin{vmatrix}1 & 1\\1 & 0\end{vmatrix} = -1$$

$$\text{So cofactor of matrix of } A = \begin{bmatrix}-2 & 5 & 1\\0 & -2 & 2\\2 & -1 & -1\end{bmatrix}$$

$\because$ the trnspose of cofactor matrix A is adj (A)

$$\text{So adj }(A) = \begin{bmatrix}-2 & 0 & 2\\5 & -2 & -1\\1 & 2 & -1\end{bmatrix}$$

$|A| = 1(0-2) - 1(1-6) + 1(1-0)$

$= -2 + 5 + 1$

$= 4$

& $A^{-1} = \frac{1}{|A|}$ adj (A)

$$\text{so, } A^{-1} = \frac{1}{4}\begin{bmatrix}-2 & 0 & 2\\5 & -2 & -1\\1 & 2 & -1\end{bmatrix}$$

Now, the given system of eq^n is

$x + y + z = 6$
$x + 2z = 7$
$3x + y + z = 12$
Writing the above equation in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

$$or, \quad \frac{1}{4} \begin{bmatrix} -12 + 0 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix}$$

$$or, \quad \frac{1}{4} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$or, \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$i.e\ x = 3,\ y = 1,\ z = 2$

OR

We know that
A = I A

$$\text{or, } \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A} \text{ [Applying } R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A} \text{ [Applying } R_2 \to R_2 + 2R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \mathbf{A} \text{ [Applying } R_1 \to R_1 + (-2)R_2, R_3 \to R_3 + 2R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \mathbf{A} \text{ [Applying } R_1 \to R_1 + 2R_3]$$

$$\text{Hence, } \mathbf{A}^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

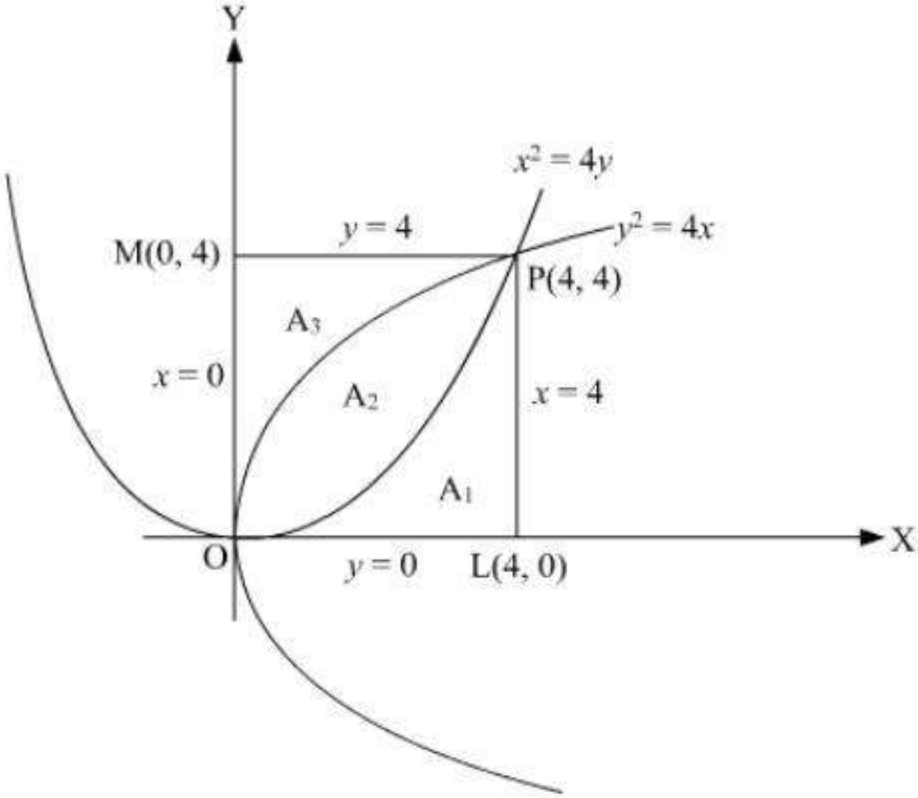
Question 26

Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by sides $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

OR

Using integration, find the area of the triangle whose vertices are (2, 3), (3, 5) and (4, 4).

SOLUTION:



A_1, A_2, A_3 are the areas denoted in the figure. We need to prove $A_1 = A_2 = A_3$.

$$A_1 = \int_0^4 y_1 \, \mathrm{d}x$$
$$= \int_0^4 \frac{x^2}{4} \, \mathrm{d}x$$
$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4$$
$$= \frac{16}{3} \text{ sq. units}$$
$$A_2 = \int_0^4 (y_2 - y_1) \, \mathrm{d}x$$
$$= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) \mathrm{d}x$$
$$= \left[\frac{4}{3}x^{\frac{3}{2}} - \frac{x^3}{12}\right]_0^4 = \frac{16}{3} \text{sq. units}$$
$$A_3 = \text{area bounded by } y^2 = 4x,\ y = 0 \text{ and } y = 4$$

$= \int_0^4 x_1 \, \mathrm{d}y$

$= \int_0^4 \frac{y^2}{4} \, \mathrm{d}y$

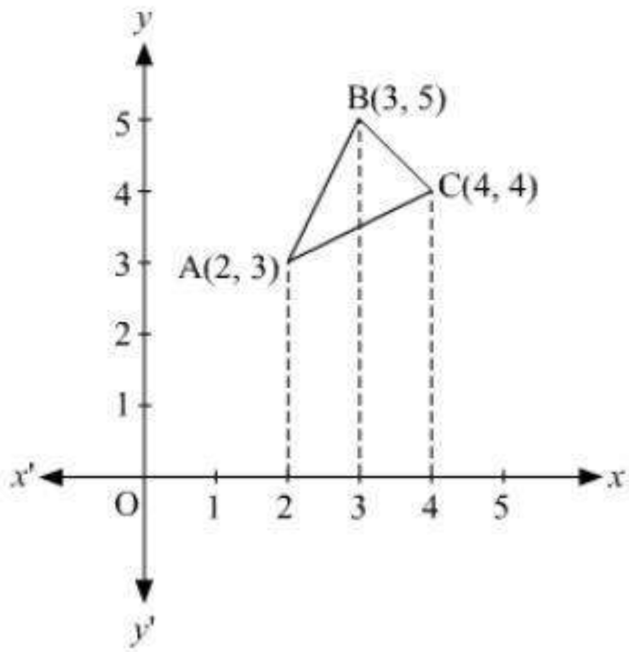
$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{16}{3}$

Therefore, $A_1 = A_2 = A_3 = \frac{16}{3}\text{sq. units}$

Thus, $y^2 = 4x$ and $x^2 = 4y$ divide the area of square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts.

OR

The Vertices of ΔABC are A (2, 3), B (3, 5), and C (4, 4)



Equation of line segment AB is

$(y-5) = \frac{5-3}{3-2}(x-3)$

$= y - 5 = 2(x-3)$

$= y = 2x - 1$

Equation of line segment BC is

$(y-5) = \frac{5-4}{3-4}(x-3)$

$= y - 5 = -1(x-3)$

$= y = -x + 8$

Equation of line segment AC is

$$(y-4)=\frac{4-3}{4-2}(x-4)$$
$$=y-4=\frac{1}{2}(x-4)$$
$$=y=\frac{x}{2}+2$$
$$\therefore \text{ Area of } \Delta \text{ ABC}=\int_2^3\left[(2x-1)-\left(\frac{x}{2}+2\right)\right]dx+\int_3^4\left[(-x+8)-\left(\frac{x}{2}+2\right)\right]\cdot dx$$
$$=\int_2^3\left(\frac{3x}{2}-3\right)\cdot dx+\int_3^4\left(\frac{-3x}{2}+6\right)\cdot dx$$
$$=\left[\frac{3x^2}{4}-3x\right]_2^3+\left[\frac{-3x^2}{4}+6x\right]_3^4$$
$$=\left(\frac{27}{4}-9\right)-(3-6)+(-12+24)-\left(\frac{-27}{4}+18\right)$$
$$=\frac{3}{2}\text{ sq. units.}$$

Question 27

A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹ 15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit.

SOLUTION:

Let x articles of model A and y articles of model B be made.

Number of articles cannot be negative.

Therefore, $x, y \geq 0$

According to the question, the making of a model A requires 2 hrs. work by a skilled man and the model B requires 1 hr by a skilled man

$2x + y \leq 40$

The making of a model A requires 2 hrs. work by a semi-skilled man model B requires 3 hrs. work by a semi-skilled man.

$2x + 3y \leq 80$

Total profit = Z = 15*x* + 10*y* which is to be maximised

Thus, the mathematical formulation of the given linear programmimg problem is

Max Z = 15*x* + 10*y*

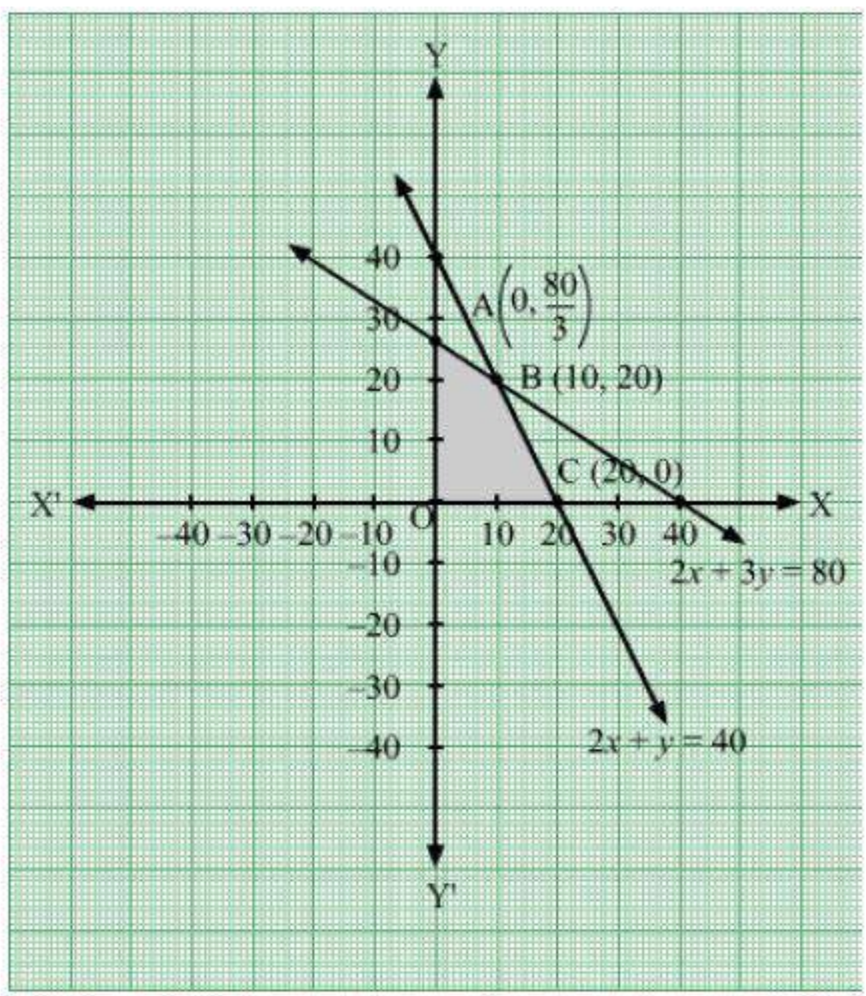
subject to

$2x + y \le 40$

$2x + 3y \le 80$

$X \ge 0$
$Y \ge 0$

The feasible region determined by the system of constraints is



The corner points are A$\left(0, \frac{80}{3}\right)$, B(10, 20), C(20, 0)

The values of Z at these corner points are as follows

Corner point	Z= 15x+10y
A	$\frac{800}{3}$
B	350
C	300

The maximum value of Z is 300 which is attained at C(20, 0)

Thus, the maximum profit is Rs 300 obtained when 10 units of deluxe model and 20 unit of ordinary model is produced.

Question 28

Find the vector and Cartesian equations of the plane passing through the points (2, 2 −1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above.

OR

Find the vector equation of the plane that contains the lines $\vec{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$ and the point (−1, 3, −4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane thus obtained.

SOLUTION:

step 1

The given points are A(2, 2, −1), B(3, 4, 2) and C(7, 0, 6)

Let $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} + 4\hat{j} + 2\hat{k}$

$\vec{c} = 7\hat{i} + 6\hat{k}$

Hence the vector equation of the plane passing through the points

$$\left(\vec{r} - \vec{a}\right) \cdot \left(\overrightarrow{AB} \times \overrightarrow{AC}\right) = 0$$

SOLUTION:

step 1

The given points are A(2, 2, −1), B(3, 4, 2) and C(7, 0, 6)

$$\text{Let } \vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$$

$$\vec{b} = 3\hat{i} + 4\hat{j} + 2\hat{k}$$

$$\vec{c} = 7\hat{i} + 6\hat{k}$$

Hence the vector equation of the plane passing through the points

$$\left(\vec{r} - \vec{a}\right) \cdot \left(\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}\right) = 0$$

$$= \left(\vec{r} - \vec{a}\right) \cdot \left(\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right)\right) = 0$$

Now

$$\vec{b} - \vec{a} = \left(3\hat{i} + 4\hat{j} + 2\hat{k}\right) - \left(2\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$\Rightarrow \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{c} - \vec{a} = \left(7\hat{i} + 6\hat{k}\right) - \left(2\hat{i} + 2\hat{j} - \hat{k}\right)$$

$$= 5\hat{i} - 2\hat{j} + 7\hat{k}$$

So the required vector equation of plane is

$$\left[\vec{r} - \left(2\hat{i} + 2\hat{j} - \hat{k}\right)\right] \cdot \left[\left(\hat{i} + 2\hat{j} + 3\hat{k}\right) \times \left(5\hat{i} - 2\hat{j} + 7\hat{k}\right)\right] = 0$$

Step 2

$$\left(\vec{b} - \vec{a}\right) \times \left(\vec{c} - \vec{a}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= \hat{i}\,(14 + 6) - \hat{j}\,(7 - 15) + \hat{k}\,(-2 - 10)$$

$$= 20\hat{i} + 8\hat{j} - 12\hat{k}$$

$$\Rightarrow \left(\vec{r} - \left(2\hat{i} + 2\hat{j} - \hat{k}\right)\right) \cdot \left(20\hat{i} + 8\hat{j} - 12\hat{k}\right) = 0$$

$$\left(\vec{r} - \left(2\hat{i} + 2\hat{j} - \hat{k}\right)\right) \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 0$$

$$\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = \left(2\hat{i} + 2\hat{j} - \hat{k}\right) \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right)$$

$$\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 10 + 4 + 3$$

$$\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 17$$

This is the required vector equation of the plane

Step 3

The Cartesian Equation of the plane passing through the three points is given as below-

$5x + 2y - 3z - 17 = 0$

This is required cartesian equation of the plane.

The equation of plane parallel to $5x + 2y - 3z - 17 = 0$ will be $5x + 2y - 3z + \lambda = 0$

$\therefore$ it passes through (4, 3, 1).

So, $5 \times 4 + 2 \times 3 - 3 \times 1 + \lambda = 0$

$20 + 6 - 3 + \lambda = 0$

So, $\lambda = -23$

so the equation of the plane will be

$5x + 2y - 3z - 23 = 0$

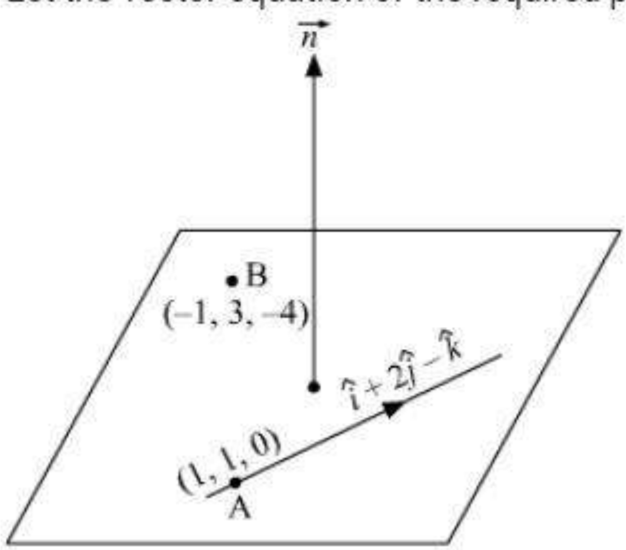
$5x + 2y - 3z = 23$

so the vector form of the equation of plane will be

$$\vec{r} \cdot \left(5\hat{i} + 2\hat{j} - 3\hat{k}\right) = 23$$

OR

Let the vector equation of the required plane be $\vec{r} \cdot \vec{n} = d$



The plane contains the line $\vec{r} = \hat{i} + \hat{j} + \lambda\left(\hat{i} + 2\hat{j} - \hat{k}\right)$

Since the plane passes through point A and B. So $\overrightarrow{n}$ will be parallel to vector

$$\overrightarrow{\mathrm{AB}} \times \left(\hat{i}+2\hat{j}-\hat{k}\right)$$

$$\overrightarrow{\mathrm{AB}} = \overrightarrow{\mathrm{OB}} - \overrightarrow{\mathrm{OA}}$$

$$= \left(-\hat{i}+3\hat{j}-4\hat{k}\right) - \left(\hat{i}+\hat{j}\right)$$

$$= -2\hat{i}+2\hat{j}-4\hat{k}$$

$$\overrightarrow{\mathrm{AB}} \times \left(\hat{i}+2\hat{j}-\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix}$$

$$= \hat{i}(-8+2) - \hat{j}(-4-2) + \hat{k}(2+4)$$

$$= -6\hat{i}+6\hat{j}+6\hat{k}$$

which is a normal vector to the plane.

So the equation of plane will be $\overrightarrow{r} \cdot \left(-6\hat{i}+6\hat{j}+6\hat{k}\right) = d$

$\therefore$ it passes through (1, 1, 0) so $\left(\hat{i}+\hat{j}\right) \cdot \left(-6\hat{i}+6\hat{j}+6\hat{k}\right) = d$ or, $d = 0$

equation of plane is $\overrightarrow{r} \cdot \left(-6\hat{i}+6\hat{j}+6\hat{k}\right) = 0$

$$\overrightarrow{r}\left(\hat{i}-\hat{j}-\hat{k}\right) = 0$$

in Cartesian plane,

$$\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \cdot \left(\hat{i}-\hat{j}-\hat{k}\right) = 0$$

$x - y - z = 0$

So, the perpendicular distance of the plane from the point (2, 1, 4) is

$$= \left|\frac{2-1-4}{\sqrt{1^2+(-1)^2+(-1)^2}}\right| = \left|\frac{-3}{\sqrt{3}}\right| = \sqrt{3}\,\text{unit.}$$

Question 29

Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings.

SOLUTION:

Let X denote the number of kings in a sample of 2 cards drawn from a well-shuffled pack of 52 playing cards. Then, X can take the values 0, 1 and 2.

Now,

$$P(X=0)$$
$$=P(\text{no king})$$
$$=\frac{^{48}C_2}{^{52}C_2}$$
$$=\frac{1128}{1326}$$
$$=\frac{188}{221}$$

$$P(X=1)$$
$$=P(1\ \text{king})$$
$$=\frac{^{4}C_1\times{}^{48}C_1}{^{52}C_2}$$
$$=\frac{192}{1326}$$
$$=\frac{32}{221}$$

$$P(X=2)$$
$$=P(2\ \text{kings})$$
$$=\frac{^{4}C_2}{^{52}C_2}$$
$$=\frac{6}{1326}$$
$$=\frac{1}{221}$$

Thus, the probability distribution of X is given by

X	$P(X)$
0	$\frac{188}{221}$
1	$\frac{32}{221}$
2	$\frac{1}{221}$

Computation of mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	$\frac{188}{221}$	0	0
1	$\frac{32}{221}$	$\frac{32}{221}$	$\frac{32}{221}$
2	$\frac{1}{221}$	$\frac{2}{221}$	$\frac{4}{221}$
		$\sum p_i x_i = \frac{34}{221}$	$\sum p_i x_i^2 = \frac{36}{221}$

$$\text{Mean} = \sum p_i x_i = \frac{34}{221}$$

$$\text{Variance} = \sum p_i {x_i}^2 - (\text{Mean})^2$$

$$= \frac{36}{221} - \left(\frac{34}{221}\right)^2$$

$$= \frac{7956-1156}{48841}$$

$$= \frac{6800}{48841}$$

$$= \frac{400}{2873}$$