* Differential Equations *

Q.1 Solve $(\tan^{-1}x - y)\,dx = (1+x^2)\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{\tan^{-1}x - y}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\tan^{-1}x}{1+x^2} - \frac{y}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{1+x^2}\right) y = \frac{\tan^{-1}x}{1+x^2}$$

[Linear DE of form $\frac{dy}{dx} + Py = Q$]

Here, $P = \frac{1}{1+x^2}$, $Q = \frac{\tan^{-1}x}{1+x^2}$

Integrating factor

$$IF = e^{\int P\,dx}$$
$$= e^{\int \frac{dx}{1+x^2}}$$
$$= e^{\tan^{-1}x}$$

$\therefore$ Solution $y(IF) = \int Q(IF)\,dx + c$

$$\Rightarrow y e^{\tan^{-1}x} = \int \frac{\tan^{-1}x}{1+x^2} e^{\tan^{-1}x} + c$$

$$\Rightarrow y e^{\tan^{-1}x} = I + c \quad \text{—— (1)} \quad \left[\text{let } I = \int \frac{\tan^{-1}x}{1+x^2} e^{\tan^{-1}x}\right]$$

for $I \Rightarrow$ let $\tan^{-1}x = t$

$$\Rightarrow \frac{1}{1+x^2} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dx}{1+x^2} = dt$$

$\therefore I = \int t e^t dt$

$= t \int e^t dt - \int \left[\frac{d}{dx}(t) \int e^t dt \right] dt$

$= te^t - \int e^t dt$

$= te^t - e^t + c_1$

$= \tan^{-1}x \, e^{\tan^{-1}x} - e^{\tan^{-1}x} + c_1$

Now, (i) $\Rightarrow y e^{\tan^{-1}x} = \tan^{-1}x \, e^{\tan^{-1}x} - e^{\tan^{-1}x} + c$

$\Rightarrow y = \tan^{-1}x - 1 + \frac{c}{e^{\tan^{-1}x}}$

$\boxed{\text{Q.2 } e^{x-y}dx + e^{y-x}dy = 0}$

$\Rightarrow \frac{dy}{dx} = \frac{-e^{x-y}}{e^{y-x}}$

$\Rightarrow \frac{dy}{dx} = - \frac{\frac{e^x}{e^y}}{\frac{e^y}{e^x}}$

$\Rightarrow \frac{dy}{dx} = - \frac{e^x}{e^y} \times \frac{e^x}{e^y}$

$\Rightarrow \frac{dy}{dx} = - \frac{e^{2x}}{e^{2y}}$

$\Rightarrow \frac{dy}{dx} = - \frac{e^{2x}}{e^{2y}}$

$\Rightarrow e^{2y} dy = -e^{2x} dx$

$\Rightarrow \frac{e^{2y}}{2} = -\frac{e^{2x}}{2} + c_1$

$\Rightarrow e^{2y} + e^{2x} = 2c_1$

$\Rightarrow e^{2y} + e^{2x} = c$ //

Q.3 Find the particular solution of the differential Equation
$\frac{dy}{dx} + 2y\tan x = \sin x$; $\boxed{y=0, \text{ when } x=\pi/3}$

Sol^n $\frac{dy}{dx} + (2\tan x)\, y = \sin x$ From $\left[\frac{dy}{dx} + Py = Q\right]$

Here $P = 2\tan x$ $Q = \sin x$

Integrating factor.

$IF = e^{\int P dx}$

$= e^{\int 2\tan x dx}$

$= e^{2\log(\sec x)}$

$= e^{\log(\sec x)^2}$

$= \sec^2 x$

$\therefore$ Solution,

$y(IF) = \int Q.(IF)\, dx + c$

$\Rightarrow y\sec^2 x = \int \sin x \sec^2 x\, dx + c$

$\Rightarrow y\sec^2 x = \int \frac{\sin x\, dx}{\cos^2 x} + c$

$\Rightarrow y\sec^2 x = I + c_1$ [Let] —— (1)

Let $\cos x = t$

$\Rightarrow -\sin x dx = dt$

$y\sec^2 x = -\int \frac{dt}{t^2} + c_1$

$\Rightarrow y\sec^2 x = -\frac{t^{-2+1}}{-2+1} + c$

$\Rightarrow y\sec^2 x = +t^{-1} + c$

$\Rightarrow y\sec^2 x = \frac{+1}{\cos x} + c$

$\Rightarrow y\sec^2 x = +\sec x + c$ —— (II) //

when, $x = \pi/3$, $y = 0$

(II) $\Rightarrow 0 = +\sec \pi/3 + c$

$\Rightarrow \boxed{c = -2}$

$\therefore$ (11) $\Rightarrow$ $y \sec^2 x = \sec x - 2$

Q.4. Find the solution curve passing through the point $(1,-1)$ of the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$

Sol^n

$$xy\frac{dy}{dx} = (x+2)(y+2)$$

$$\Rightarrow \frac{y\,dy}{y+2} = \frac{(x+2)\,dx}{x}$$

$$\Rightarrow \frac{(y+2-2)\,dy}{y+2} = dx + 2\frac{dx}{x}$$

Integrating both side.

$$\int\left(\frac{y+2-2}{y+2}\right)dy = \int dx + 2\int\frac{dx}{x}$$

$$\Rightarrow \int dy - 2\int\frac{dy}{y+2} = \int dx + 2\int\frac{dx}{x}$$

$$\Rightarrow y - 2\log|y+2| = x + 2\log x + c \quad \text{——— (1)}$$

When $x = 1$, $y = -1$

$$(1) \Rightarrow -1 - 2\log(-1+2) = 1 + \log 1 + c$$

$$\Rightarrow -1 - 2\log 1 = 1 + 0 + c$$

$$\Rightarrow c = -2$$

Putting $c = -2$ in (1)

$$\Rightarrow y - 2\log|y+2| = x + 2\log x - 2$$ 4

$$\boxed{5.Q.\quad \left(1+e^{\frac{x}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0}$$

$$\Rightarrow e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = -\left(1+e^{\frac{x}{y}}\right)dx$$

$$\Rightarrow \frac{dx}{dy} = \frac{-e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)}{1+e^{\frac{x}{y}}}$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^{\frac{x}{y}}\left(\frac{x}{y}-1\right)}{1+e^{\frac{x}{y}}} \quad \text{---} \quad (1)$$

As it is homogeneous D.E.

let $\frac{x}{y} = v$

$\Rightarrow x = vy$

Differentiating both side w.r.t y.

$$\frac{dx}{dy} = \frac{dv}{dy}(y) + v$$

Substituting the values in (1) $\Rightarrow$

$$\frac{dv}{dy}(y) + v = \frac{e^{v}(v-1)}{1+e^{v}}$$

$$\Rightarrow \left(\frac{dv}{dy}\right)y = \frac{e^{v}(v-1)}{1+e^{v}} - v$$

$$\Rightarrow y\left(\frac{dv}{dy}\right) = \frac{\cancel{ve^{v}} - e^{v} - v - \cancel{ve^{v}}}{1+e^{v}}$$

$$\Rightarrow y\left(\frac{dv}{dy}\right) = \frac{-(v+e^v)}{1+e^v}.$$

$$\Rightarrow \left(\frac{1+e^v}{v+e^v}\right) dv = -\frac{dy}{y} \quad \text{—— (1)}$$

Let $v+e^v = t$

$$\Rightarrow (1+e^v) = \frac{dt}{dv}$$

$$\Rightarrow (1+e^v)\,dv = dt$$

$$(1) \Rightarrow \int \frac{(1+e^v)\,dv}{v+e^v} = -\int \frac{dy}{y}$$

$$\Rightarrow \int \frac{dt}{t} = -\int \frac{dy}{y}$$

$$\Rightarrow \log t = -\log y + \log c$$

$$\Rightarrow \log(v+e^v) = \log \frac{c}{y}$$

$$\Rightarrow v+e^v = \frac{c}{y}$$

$$\Rightarrow \frac{x}{y} + e^{\frac{x}{y}} = \frac{c}{y}$$

$$\Rightarrow x + ye^{x/y} = c$$

Q.6. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\frac{x^2+y^2}{2xy}$ is given by $x^2 - y^2 = cx$.

Sol^n

A/Q. $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$

$$\cancel{\Rightarrow \frac{dy}{dx} = \frac{x^2}{2xy} + \frac{y^2}{2xy}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{x^2+y^2}{x^2}}{\frac{2xy}{x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+\left(\frac{y}{x}\right)^2}{2\frac{y}{x}} \quad \text{——— (A)}$$

It is homogeneous D.E.

Let $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

(A) $\Rightarrow x\frac{dv}{dx} + v = \frac{1+v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$

$x\frac{dv}{dx} = \frac{1-v^2}{2v}$

$\Rightarrow \frac{2v\,dv}{1-v^2} = \frac{dx}{x}$

Integrating both side.

$-\int \frac{-2v\,dv}{1-v^2} = \int \frac{dx}{x}$

$-\log(1-v^2) = \log x + \log$

$\Rightarrow \frac{1}{1-v^2} = c_1 x$

$$\Rightarrow \frac{1}{1-\left(\frac{y}{x}\right)^2} = c_1 x$$

$$\Rightarrow \frac{1}{\frac{x^2-y^2}{x^2}} = c_1 x$$

$$\Rightarrow \frac{x^2}{x^2-y^2} = c_1 \cancel{x}$$

$$\Rightarrow x^2-y^2 = \frac{x}{c_1}$$

$$\Rightarrow x^2-y^2 = cx \quad \left[\text{let } \frac{1}{c_1} = c\right]$$

Hence Proved //

Q.7. $(xdy - ydx)y\sin\left(\frac{y}{x}\right) = (ydx + xdy)x\cos\left(\frac{y}{x}\right)$

$$\Rightarrow xy\sin\left(\frac{y}{x}\right)dy - y^2\sin\left(\frac{y}{x}\right)dx = xy\cos\left(\frac{y}{x}\right)dx + x^2\cos\left(\frac{y}{x}\right)dy$$

$$\Rightarrow xy\sin\left(\frac{y}{x}\right)dy - x^2\cos\left(\frac{y}{x}\right)dy = xy\cos\left(\frac{y}{x}\right)dx + y^2\sin\left(\frac{y}{x}\right)dx$$

$$\Rightarrow \left[xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)\right]dy = \left[xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)\right]dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{xy\cos\left(\frac{y}{x}\right) + y^2\sin\left(\frac{y}{x}\right)}{xy\sin\left(\frac{y}{x}\right) - x^2\cos\left(\frac{y}{x}\right)} \quad \text{——— (A)}$$

~~Let $\frac{y}{x} = v$~~

~~$\Rightarrow y = vx$~~

~~$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$~~

~~Substituting the values in (A)~~

~~(A) ⇒ $x\frac{dv}{dx} + v =$~~

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{\cancel{x}y\cos\left(\frac{y}{x}\right)}{x^{\cancel{2}}} - \frac{y^2}{x^2}\sin\left(\frac{y}{x}\right)}{\frac{\cancel{x}y\sin\left(\frac{y}{x}\right)}{x^{\cancel{2}}} - \frac{\cancel{x^2}}{\cancel{x^2}}\cos\left(\frac{y}{x}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}\cos\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2\sin\left(\frac{y}{x}\right)}{\frac{y}{x}\sin\left(\frac{y}{x}\right) - \cos\left(\frac{y}{x}\right)} \quad \text{—— (A)}$$

It is homogeneous D.E.

let $\frac{y}{x} = v$

$\Rightarrow y = vx$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$

Substituting the values in (A) ⇒

$$x\frac{dv}{dx} + v = \frac{v\cos v - v^2\sin v}{v\sin v - \cos v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v - v^2\sin v}{v\sin v - \cos v} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v - \cancel{v^2\sin v} - \cancel{v^2\sin v} + v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \frac{v\sin v - \cos v}{2v\cos v}\,dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{2}\tan v - \frac{1}{2v}\right)dv = \frac{dx}{x}$$

Integrating both side

$$\Rightarrow \frac{1}{2}\int \tan v\, dv - \frac{1}{2}\int \frac{dv}{v} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\log \sec v - \frac{1}{2}\log v = \log x + \log c$$

$$\Rightarrow \log \sec^2 v - \log v^2 = \log x + \log c$$

$$\Rightarrow \log \frac{\sec^2 v}{v^2} = \log cx$$

$$\Rightarrow \frac{\sec^2 \frac{y}{x}}{\frac{y^2}{x^2}} = cx$$

$$\Rightarrow \sec^2 \frac{y}{x} = cx \frac{y^2}{x^2}$$

$$\Rightarrow x \sec^2 \frac{y}{x} = cy^2$$

Q.8. Show that the DE $2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$ is homogeneous. Also find the particular solution given that $x = 0$, $\because y = 1$.

Sol^n $2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$

$\Rightarrow 2ye^{\frac{x}{y}}dx = \left(2xe^{\frac{x}{y}} - y\right)dy$

$\Rightarrow \frac{dx}{dy} = \frac{2xe^{x/y} - y}{2ye^{x/y}}$

$\Rightarrow \frac{dx}{dy} = \frac{\frac{2xe^{x/y} - y}{y}}{\frac{2ye^{x/y}}{y}}$

$\Rightarrow \frac{dx}{dy} = \frac{2\left(\frac{x}{y}\right)e^{\left(\frac{x}{y}\right)} - 1}{2e^{\left(\frac{x}{y}\right)}}$ —— (A)

As $\frac{dx}{dy} = f(x,y) = \lambda^n F\left(\frac{x}{y}\right)$

So it is homogeneous. D.E

let $\frac{x}{y} = v$

$\Rightarrow x = vy$

$\Rightarrow \frac{dx}{dy} = y\frac{dv}{dy} + v$

Substituting the values in (A)

$$\Rightarrow y\frac{dv}{dy} + v = \frac{2ve^{v} - 1}{2e^{v}}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{2ve^{v} - 1}{2e^{v}} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{\cancel{2ve^{v}} - 1 - \cancel{2ve^{v}}}{2e^{v}}$$

$$\Rightarrow 2e^{v}dv = -\frac{dy}{y}$$

Integrating both side.

$$\int 2e^{v}dv = -\int\frac{dy}{y}$$

$$\Rightarrow 2e^{v} = -\log y + c$$

$$\Rightarrow 2e^{\frac{x}{y}} = -\log y + c \quad \text{——— (1)}$$

When $x = 0,\ y = 1$

$$\Rightarrow 2e^{0} = -\log 1 + c$$

$$\Rightarrow 2 = 0 + c$$

$$\Rightarrow c = 2$$

$$(1) \Rightarrow 2e^{\frac{x}{y}} = -\log y + 2$$

$$\Rightarrow 2e^{\frac{x}{y}} + \log y = 2$$

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