Q.10. $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$ —— (i)

2017 AHSEC

$= \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1+\cos^2(\pi - x)} dx$

[Property. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$]

$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x} dx$ —— (ii)

(i) + (ii) ⇒

$2I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx + \int_0^{\pi} \frac{(\pi - x) \sin x}{1+\cos^2 x} dx$

$\Rightarrow 2I = \int_0^{\pi} \frac{(\cancel{x} + \pi - \cancel{x}) \sin x}{1+\cos^2 x} dx$

$\Rightarrow 2I = \int_0^{\pi} \frac{\pi \sin x \, dx}{1+\cos^2 x}$

$\Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x \, dx}{1+\cos^2 x}$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

[when, $x = 0$, $t = \cos 0 = 1$
$x = \pi$, $t = \cos \pi = -1$]

$I = \frac{\pi}{2} \int_1^{-1} \frac{-dt}{1+t^2}$

$= \frac{\pi}{2} \int_{-1}^{1} \frac{dt}{1+t^2}$

[Property: $\int_a^b f(x) dx = -\int_b^a f(x) dx$]

$= \frac{\pi}{2} \left[\tan^{-1} t\right]_{-1}^{1}$

$= \frac{\pi}{2} \left[\tan^{-1} 1 - \tan^{-1}(-1)\right]$

$= \frac{\pi}{2} \left[\frac{\pi}{4} + \frac{\pi}{4}\right]$

$= \frac{\pi}{2} \times \frac{\pi}{2}$

$= \frac{\pi^2}{4}$

Q. 13. 2017 AHSBS

$I = \int \frac{\sin 2x}{a^2\cos^2 x + b^2 \sin^2 x} dx$

let $a^2\cos^2 x + b^2\sin^2 x = t$

$\Rightarrow -2a^2 \cos x \sin x + 2b^2 \sin x \cos x = \frac{dt}{dx}$

$\Rightarrow +2 \sin x \cos x \, (b^2 - a^2) = \frac{dt}{dx}$

$\Rightarrow \sin 2x \, dx = \frac{dt}{b^2 - a^2}$

$\therefore \int \left[\frac{dt/b^2-a^2}{t} \right]$

$= \frac{1}{b^2-a^2} \int \frac{dt}{t}$

$= \frac{1}{b^2-a^2} \log t + c$

$= \frac{1}{b^2-a^2} \log |a^2\cos^2 x + b^2 \sin^2 x| + c$

Q. 14. 2019 AHSEC

$$I = \int \frac{\cos 8x + 1}{\tan 2x - \cot 2x}\, dx$$

$$= \int \frac{\cos 8x + 1}{\frac{\sin 2x}{\cos 2x} - \frac{\cos 2x}{\sin 2x}}\, dx$$

$$= \int \frac{\cos 8x + 1}{\frac{\sin^2 2x - \cos^2 2x}{\cos 2x \sin 2x}}\, dx$$

$$= \int \frac{(\cos 2.4x + 1)\cos 2x \sin 2x}{-\cos 2.2x}\, dx$$

Formula

$\cos 2x = \cos^2 x - \sin^2 x$

$1 + \cos 2x = 2\cos^2 x$

$$= \int \frac{2\cos^{\cancel{2}} 4x \cos 2x \sin 2x}{-\cancel{\cos 4x}}\, dx$$

$$= -2\int \frac{\cos 4x \sin 2x \cos 2x}{1}\, dx$$

$$= -\int \cos 4x\,(2\sin 2x \cos 2x)\, dx$$

$$= -\frac{1}{2}\int 2\sin 4x \cos 4x\, dx$$

$$= -\frac{1}{2}\int \sin 8x\, dx$$

$$= -\frac{1}{2}\left(-\frac{\cos 8x}{8}\right) + c$$

$$= \frac{1}{16}\cos 8x + c$$

Q (18) $I = \int \frac{1}{x}\left(\frac{1-x}{\sqrt{1-x^2}}\right)dx$

$= \int \frac{1}{x\sqrt{1-x^2}}\,dx - \int \frac{\cancel{x}}{\cancel{x}\sqrt{1-x^2}}\,dx$

$= \int \frac{dx}{x\sqrt{1-x^2}} - \int \frac{dx}{\sqrt{1-x^2}}$

$= I_1 - I_2$

For I_1: let $\sqrt{1-x^2} = t \implies 1-x^2 = t^2$

$\Rightarrow x^2 = 1-t^2$

$\Rightarrow \frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x) = \frac{dt}{dx}$

$\Rightarrow \frac{-x}{\sqrt{1-x^2}} = \frac{dt}{dx}$

$\Rightarrow dx = \left(-\frac{\sqrt{1-x^2}}{x}\right)dt$

$I_1 = \int \frac{dx}{x\sqrt{1-x^2}}$

$= \int \frac{dx}{x(t)}$

$= \int \frac{1}{xt}\left(-\frac{\sqrt{1-x^2}}{x}\right)dt$

$= -\int \frac{1}{x\cancel{t}}\,\frac{\cancel{t}}{x}\,dt$

$= -\int \frac{dt}{x^2}$

$= -\int \frac{dt}{1-t^2} = -\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C_1$

$= -\frac{1}{2}\log\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| + C_1$

For I_2:

$I_2 = \int \frac{dx}{\sqrt{1-x^2}}$

$= \sin^{-1}x + C_2$

$\therefore I = I_1 - I_2$

$= -\frac{1}{2}\log\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right| - \sin^{-1}x + C$

Q. 16. $I = \int \frac{1}{1+\tan x}\,dx$

Dibrugarh pre-final 2024

$= \int \frac{1}{1+\frac{\sin x}{\cos x}}\,dx$

$= \int \frac{\cos x}{\cos x+\sin x}\,dx$

$= \frac{1}{2}\int \frac{2\cos x}{\cos x+\sin x}\,dx$

$= \frac{1}{2}\int \frac{-\sin x+\cos x+\cos x+\sin x}{\cos x+\sin x}\,dx$

$= \frac{1}{2}\int \frac{-\sin x+\cos x}{\cos x+\sin x}\,dx + \frac{1}{2}\int \frac{\cancel{(\cos x+\sin x)}}{\cancel{(\cos x+\sin x)}}\,dx$

$I = I_1 + I_2$ [Let]

$I_1 \Rightarrow \frac{1}{2}\int \frac{-\sin x+\cos x}{\cos x+\sin x}\,dx$

Let $\cos x+\sin x = t$

$\Rightarrow (-\sin x+\cos x)\,dx = dt$

$I_1 = \frac{1}{2}\int \frac{dt}{t}$

$= \frac{1}{2}\log t + c_1$

$= \frac{1}{2}\log|\cos x+\sin x| + c_1$

$I_2 \Rightarrow \frac{1}{2}\int dx$

$= \frac{1}{2}x + c_2$

$I = I_1 + I_2$

$= \frac{1}{2}\log|\cos x+\sin x| + \frac{1}{2}x + c$

(18) $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx$ Borpeta pre-final

$= \int_0^1 \tan^{-1}\left(\frac{x+x-1}{1+x(1-x)}\right) dx$

$= \int_0^1 \tan^{-1}\left[\frac{x+(x-1)}{1-x(x-1)}\right] dx$

$= \int_0^1 \left[\tan^{-1}x + \tan^{-1}(x-1)\right] dx$

$I = \int_0^1 \tan^{-1}x \, dx + \int_0^1 \tan^{-1}(x-1) \, dx$ —— (i)

$= \int_0^1 \tan^{-1}(1-x) \, dx + \int_0^1 \tan^{-1}(\cancel{1}-x-\cancel{1}) \, dx$

$= \int_0^1 \tan^{-1}(-(x-1)) \, dx + \int_0^1 \tan^{-1}(-x) \, dx$ [Property, $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$]

$I = -\int_0^1 \tan^{-1}(x-1) \, dx - \int_0^1 \tan^{-1}x \, dx$ —— (ii)

(i) + (ii) $\Rightarrow 2I = \cancel{\int_0^1 \tan^{-1}x \, dx} + \cancel{\int_0^1 \tan^{-1}(x-1) \, dx} - \cancel{\int_0^1 \tan^{-1}(x-1) \, dx} - \cancel{\int_0^1 \tan}$

$\Rightarrow 2I = 0$

$\Rightarrow \boxed{I = 0}$

Golaghat Pre-final

(19) $I = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$

$= \int \frac{\sqrt{\tan x} \sec^2 x}{\sin x \cos x \sec^2 x} dx$

$= \int \frac{\sqrt{\tan x} \sec^2 x}{\sin x \cancel{\cos x} \frac{1}{\cos^{\cancel{2}} x}} dx$

$= \int \frac{\sqrt{\tan x} \sec^2 x}{\tan x} dx$

$= \int (\tan x)^{1/2 - 1} \sec^2 x \, dx$

$= \int (\tan x)^{-1/2} \sec^2 x \, dx$

Let $\tan x = t$

$\Rightarrow \sec^2 x \, dx = dt$

$\therefore I = \int t^{-1/2} dt$

$= \frac{t^{-1/2+1}}{-1/2+1} + c$

$= 2\sqrt{t} + c$

$= 2\sqrt{\tan x} + c$

(20) $I = \int \sin^3 x \cos^2 x \, dx$ Nalbari Pre final

$= \int \sin^2 x \cos^2 x \sin x \, dx$

$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$

Let $\cos x = t$

$\Rightarrow -\sin x \, dx = dt$

$\Rightarrow \sin x \, dx = -dt$

$I = \int (1 - t^2) t^2 (-dt)$

$= \int (t^2 - t^4)(-dt)$

$= \int t^4 dt - \int t^2 dt$

$= \frac{t^5}{5} - \frac{t^3}{3} + C$

$= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C$

S.Q $I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx$ (Mis) 2014 AHSEC 4 marks

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{-(-\sin 2x)}}\, dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{-1(-1+1-2\sin x \cos x)}}\, dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-[\sin^2 x + \cos^2 x - 2\sin x \cos x]}}\, dx$

$= \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\, dx$

Let $\sin x - \cos x = t$

$\Rightarrow \cos x - (-\sin x) = \frac{dt}{dx}$

$\Rightarrow (\sin x + \cos x)\, dx = dt$

When,

$x = \frac{\pi}{6}$, $t = \sin\frac{\pi}{6} - \cos\frac{\pi}{6}$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}$

$= \frac{1-\sqrt{3}}{2}$

$x = \frac{\pi}{3}$ $t = \sin\frac{\pi}{3} - \cos\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2}$

$I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1^2 - t^2}}$

$= \boxed{\left[\sin^{-1} t\right]_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}}$

or.

$\boxed{\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(x) \text{ is even}}$

$= f(t) = \frac{1}{\sqrt{1-t^2}}$

$f(-t) = \frac{1}{\sqrt{1-t^2}} = f(t)$ $\boxed{\text{even function}}$

$I = 2\int_0^{\frac{\sqrt{3}-1}{2}} \frac{dt}{\sqrt{1^2-t^2}} = 2\left[\sin^{-1} t\right]_0^{\frac{\sqrt{3}-1}{2}}$

$= 2\left(\sin^{-1}\frac{\sqrt{3}-1}{2} - \sin^{-1} 0\right) = 2\sin^{-1}\frac{\sqrt{3}-1}{2}$

Q. 6. $\int_1^4 [|x-1| + |x-2| + |x-3|]\, dx$

2024 Nagaon pre-board

$= \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-3|\,dx$

$I = I_1 + I_2 + I_3$ [let]

$I_1 = \int_1^4 |x-1|\,dx$

$\left[|x-1| = \begin{cases} x-1 & x \geq 1 \\ -(x-1) & x < 1 \end{cases} \right.$

$= \int_1^4 (x-1)\,dx$

$= \left[\frac{x^2}{2} - x\right]_1^4$

$= \left(\frac{16}{2} - 4\right) - \left(\frac{1}{2} - 1\right) = 4 + \frac{1}{2} = \frac{9}{2}$

$I_2 = \int_1^4 |x-2|\,dx$

$\left[|x-2| = \begin{cases} x-2 & x \geq 2 \\ -(x-2) & x < 2 \end{cases} \right.$

$= \int_1^2 -(x-2)\,dx + \int_2^4 (x-2)\,dx$

Property $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= -\left[\frac{x^2}{2} - 2x\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^4$

$= -\left[2 - 4 - \frac{1}{2} + 2\right] + \left[8 - 8 - 2 + 4\right]$

$= 0 + \frac{1}{2} + 2 = \frac{5}{2}$

$I_3 = \int_1^4 |x-3|\,dx$

$= \int_1^3 -(x-3)\,dx + \int_3^4 (x-3)\,dx$

$= -\left[\frac{x^2}{2} - 3x\right]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4$

$= -\left[\frac{9}{2} - 9 - \frac{1}{2} + 3\right] + \left[8 - 12 - \frac{9}{2} + 9\right]$

$= -\frac{8}{2} + 6 + 5 - \frac{9}{2} = -\frac{17}{2} + 11 = \frac{5}{2} = \frac{5}{2}$

$\therefore I = I_1 + I_2 + I_3$

$= \frac{9}{2} + \frac{5}{2} + \frac{5}{2}$

$= \frac{9 + 5 + 5}{2}$

$= \frac{19}{2}$ //