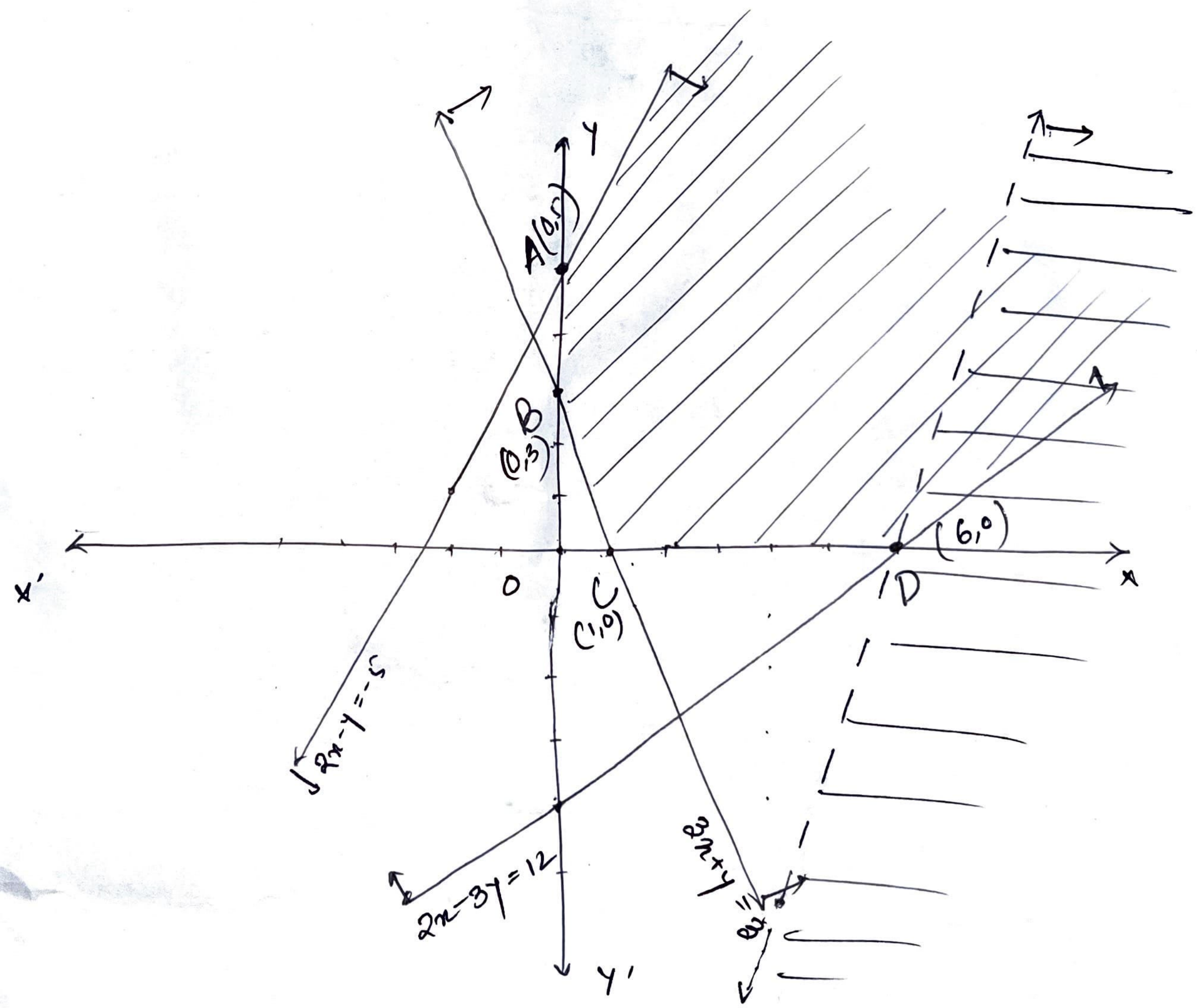
Linear Programming:

Q. Determine graphically the minimum value of the objective function $Z = -50x + 20y$ subject to the constraints.

$$2x - y \geq -5$$
$$3x + y \geq 3$$
$$2x - 3y \leq 12$$
$$x \geq 0,\ y \geq 0$$

Sol^n



$2x - y = -5$

x	0	-2
y	5	1

$3x + y = 3$

x	0	1
y	3	0

$2x - 3y = 12$

x	6	0
y	0	-4

Let $x'ox$ and yoy' be two co-ordinate axes intersecting at point O i.e at origin. We draw the lines $2x-y=-5$, $3x+y=3$, $2x-3y=12$, $x=0$. and $y=0$ on the xy plane. Taking a point $(0,0)$ on the half plane divided by the lines and check it satisfy the given inequality or ~~on~~ not.

Here, the feasible region made by the lines are unbounded and $A(0,5)$, $B(0,3)$, $C(1,0)$ and $D(6,0)$ are corner points

Corner points	$Z=-50x+20y$
$A(0,5)$	100
$B(0,3)$	60
$C(1,0)$	-50
$D(6,0)$	-300 ←

As the region is unbounded, we can't say -300 is the minimum value.

Now, $-50x+20y<-300$

$\Rightarrow -5x+2y<-30$

Draw the line $-5x+2y=-30$ on xy plane and then we found the satisfied region.

x	6	4
y	0	-5

As resulting open half Plane has common points with feasible region. then, we can say -300 is not the minimum value of Z

$\therefore$ Z has no minimum value.

Q.3 For any two vectors $\vec{a}$ and $\vec{b}$, we always have $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$ [Triangle inequality]. Prove that

Proof: The inequality trivially holds in case either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$. Because for zero ~~vectors~~ both the sides are equal.

Pattern Classes

Now let $|\vec{a}| \neq |\vec{b}| \neq 0$

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$$ [For any vector $\vec{a}$, $\vec{a}.\vec{a} = |\vec{a}|^2$]

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2$$ [Commutative law $\vec{a}.\vec{b} = \vec{b}.\vec{a}$]

$$\leq |\vec{a}|^2 + 2|\vec{a}.\vec{b}| + |\vec{b}|^2$$ [$\because x \leq |x|, \forall x \in R$]

$$\leq |\vec{a}|^2 + 2|\vec{a}||\vec{b}| + |\vec{b}|^2$$ [Cauchy Schwartz inequality $|\vec{a}.\vec{b}| \leq |\vec{a}||\vec{b}|$]

$$= (|\vec{a}| + |\vec{b}|)^2$$

$$\therefore |\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$$ //

Q.4 Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

Solⁿ: Given, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{L.H.S} = \vec{a} \times (\vec{b} + \vec{c})$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \left[(b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$$

$$= \hat{i}\left[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2\right] - \hat{j}\left[a_1b_3 + a_1c_3 - a_3b_1 - a_3c_1\right] + \hat{k}\left[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\right]$$

$$\text{R.H.S} = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1) + \hat{i}(a_2c_3 - a_3c_2) - \hat{j}(a_1c_3 - a_3c_1) + \hat{k}(a_1c_2 - a_2c_1)$$

$$= \hat{i}(a_2b_3 - a_3b_2 + a_2c_3 - a_3c_2) - \hat{j}(a_1b_3 - a_3b_1 + a_1c_3 - a_3c_1) + \hat{k}(a_1b_2 - a_2b_1 + a_1c_2 - a_2c_1)$$

$\therefore$ L.H.S = R.H.S

$\therefore \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

Q.5 Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$.

Proof: Let $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = |\vec{a}|^2 + |\vec{b}|^2$ is true

$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{b}.\vec{a} + \vec{b}.\vec{b} = |\vec{a}|^2 + |\vec{b}|^2$

$\Rightarrow \cancel{|\vec{a}|^2} + \vec{a}.\vec{b} + \vec{a}.\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2}$ [commutative law, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$]

$\therefore \Rightarrow 2\vec{a}.\vec{b} = 0$

$\Rightarrow \vec{a}.\vec{b} = 0$

Pattern classes

$\therefore \vec{a}$ and $\vec{b}$ are perpendicular.

Q.6 If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, Show that the vector $\vec{a}+\vec{b}+\vec{c}$ equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$.

Proof: $\because$ $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular vectors, we have.

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = 0 \quad , \quad \text{Given that } |\vec{a}| = |\vec{b}| = |\vec{c}|$$

let $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}, \vec{b}$ and $\vec{c}$ at angles θ_1, θ_2 and θ_3 respectively.

$\therefore$ ~~Direction cosines~~

$$\cos\theta_1 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{a}|} \qquad \left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \cos\theta\right]$$

$$= \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{a}|}$$

$$= \frac{|\vec{a}|^{\cancel{2}}}{|\vec{a}+\vec{b}+\vec{c}|\,\cancel{|\vec{a}|}}$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}}{|(\vec{a}+\vec{b}+\vec{c})|\,|\vec{b}|}$$

$$= \frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{b}|}$$

$$= \frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{b}|}$$

$$= \frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{c}|}$$

$$= \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}|\,|\vec{c}|}$$

$$= \frac{|\vec{c}|^{\cancel{2}}}{|\vec{a}+\vec{b}+\vec{c}|\,\cancel{|\vec{c}|}}$$

Pattern classes

$$= \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$\because |\vec{a}| = |\vec{b}| = |\vec{c}|$

$\therefore \cos\theta_1 = \cos\theta_2 = \cos\theta_3$

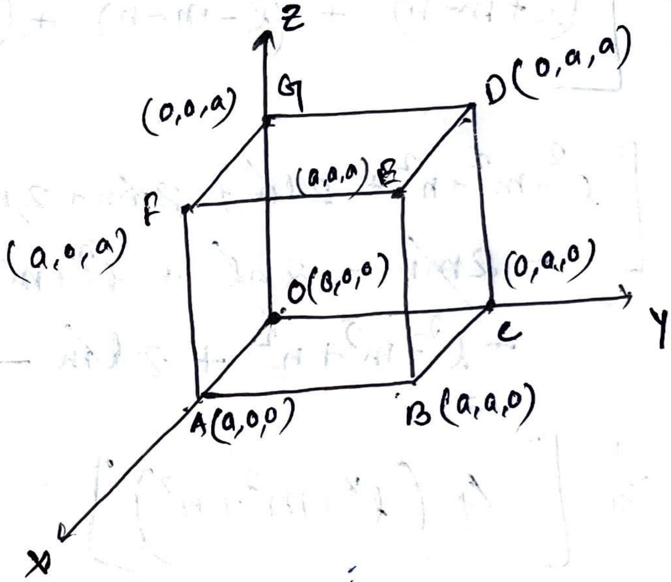
$\Rightarrow \theta_1 = \theta_2 = \theta_3$

$\therefore \vec{a}+\vec{b}+\vec{c}$ equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$

8. A line makes angles α, β, γ and δ with the diagonals of a cube, prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

<u>Soln</u> Let OABCDEFG be a cube with each side a.



The diagonals of the cube are OE, CF, AD, BG.

The D.C of the diagonal OE

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

$$= \frac{a}{\sqrt{3a^2}}, \frac{a}{\sqrt{3a^2}}, \frac{a}{\sqrt{3a^2}}$$

$$= \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Similarly, DC's of CF, AD and BG are.

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \quad \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \quad \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$$

Let l, m, n be the D.C of the given line which makes $\alpha, \beta, \gamma, \delta$ with OE, CF, AD and BG.

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n)$$

$$\cos\beta = \frac{1}{\sqrt{3}}(l-m+n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(l, m, -n)$$

[As. $\cos\alpha = (l_1 l_2 + m_1 m_2 + n_1 n_2)$

Squaring and adding,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l+m+n)^2 + (l-m+n)^2 + (-l+m+n)^2 + (l+m-n)^2\right]$$

$$= \frac{1}{3}\left[l^2+m^2+n^2 + \cancel{2lm} + \cancel{2mn} + \cancel{2nl} + l^2+m^2+n^2 - \cancel{2lm} - \cancel{2mn} + \cancel{2nl} + l^2+m^2+n^2 - \cancel{2lm} + \cancel{2mn} - \cancel{2nl} + l^2+m^2+n^2 + \cancel{2lm} - \cancel{2mn} - \cancel{2nl}\right]$$

$$= \frac{1}{3}\left[4(l^2+m^2+n^2)\right]$$

$$= \frac{1}{3} \times (4 \times 1) \qquad [l^2+m^2+n^2=1]$$

$$= \frac{4}{3}$$

H.P.

* Important Questions
Veetor and 3-dimensional Geometrey

Ex 10·2 : * Property 2 proof (Associative law) old book page 431

* Example 7, 9, 12
* Exercise 15, 17

Ex 10·3 : * Example 17, 18, 20 (proof)

* Exercise 6, 9, 13, 17

Ex 10·4 * Example 24, 25

* Exercise 2, 3, 7.

Miscellaneous * Example 28,

* Exercise 12, 14, 15,

Ex 11·1 : * Example 2

* Exercise 3

Ex 11·2 *Shortest distance between skew lines

* Exercise 14 – 17 "

Chapter 11 Miscellaneous

Example 1 VVI

Exercise 20, 6