Application of Derivatives

1. Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (i) Increasing (ii) Decreasing.

Sol^n:

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$f'(x) = 6x^2 - 6x - 36 = 0$$

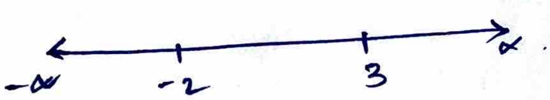
$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow x^2 - 3x + 2x - 6 = 0$$

$$\Rightarrow x(x-3) + 2(x-3) = 0$$

$$\Rightarrow (x-3)(x+2) = 0$$

$$\therefore x = 3, -2 \quad [\text{critical point}]$$



In the interval $(-\infty, -2)$

let $x = -3$

$$f'(x) = 6(-3)^2 - 6(-3) - 36 = 54 + 18 - 36 = 36 > 0$$

$$\begin{array}{r}16\\6\\\hline 96\end{array}$$

In the interval $(-2, 3)$

let $x = 0$

$$f'(0) = 6(0)^2 - 6(0) - 36 = -36 < 0$$

In the interval $(3, \infty)$

let $x = 4$

$$f'(4) = 6(4)^2 - 6 \times 4 - 36 = 96 - 24 - 36 = 96 - 60 = 36 > 0$$

As in the interval $(-\infty, -2)$ and $(3, \infty)$ $f'(x)$ is +ve

$\therefore$ function is increasing

As in the interval $(-2, 3)$ $f'(x)$ is −ve.

f^n is decreasing.

Q.3. Show that the function f is given by $f(x) = \tan^{-1}(\sin x + \cos x)$, $x > 0$ is always a strictly increasing function in $(0, \pi/4)$

Sol^n

$$f(x) = \tan^{-1}(\sin x + \cos x)$$

$$f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} (\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{1 + \sin^2 x + \cos^2 x + 2\sin x \cos x}$$

$$= \frac{\cos x - \sin x}{1 + 1 + \sin 2x}$$

$$f'(x) = \frac{\cos x - \sin x}{2 + \sin 2x}$$

for $x \in (0, \frac{\pi}{4})$ $\cos x > \sin x$

so $\cos x - \sin x$ is always +ve

We know, $-1 \leq \sin 2x \leq 1$

$$\Rightarrow -1 + 2 \leq \sin 2x + 2 \leq 1 + 2$$

$$\Rightarrow 1 \leq 2 + \sin 2x \leq 3$$

$\therefore$ $2 + \sin 2x$ is also positive.

$\therefore$ overall $f'(x) = \dfrac{\cos x - \sin x}{2 + \sin 2x} > 0$

$\therefore$ the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$

Q.4. The total revenue recieved from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, what is the marginal revenue when $x = 5$.

Sol^n Maregional revenue.

$$MR = \frac{dR(x)}{dx}$$

$$= \frac{d}{dx}(3x^2 + 36x + 5)$$

$$MR = 6x + 36$$

$\therefore$ MR at $x = 5$

$$MR|_{x=5} = 6 \times 5 + 36 = 66$$

Q.5. Find the intervals in which the function f given by $f(x) = \sin 3x$, $x \in [0, \pi/2]$ is
(a) increasing (b) Decreasing.

Sol^n:

$f(x) = \sin 3x$

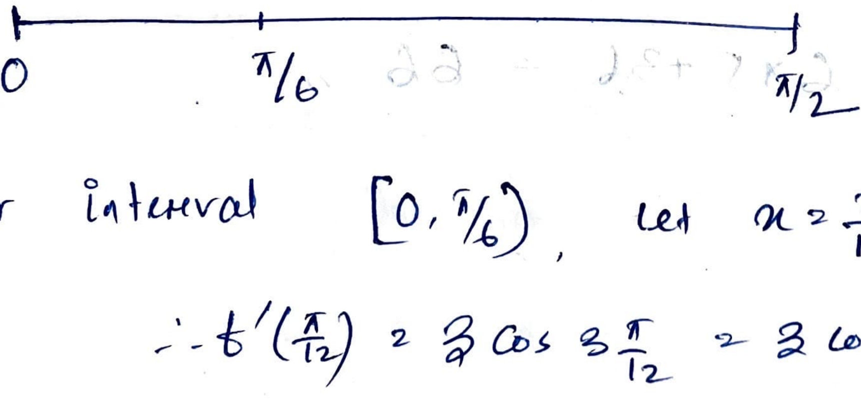
$\Rightarrow f'(x) = 3\cos 3x = 0$

$\Rightarrow \cos 3x = 0$

$\Rightarrow \cos 3x = \cos \pi/2 = \cos 3\pi/2$

$3x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$ [critical points]



For interval $[0, \pi/6)$, let $x = \frac{\pi}{12}$

$\therefore f'(\frac{\pi}{12}) = 3\cos 3\frac{\pi}{12} = 3\cos\frac{\pi}{4} = 3 \times \frac{1}{\sqrt{2}} > 0$

$\therefore$ In the interval $[0, \pi/6)$ the function is increasing.

Again, For interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$. let $x = \frac{\pi}{3}$

$f'(\frac{\pi}{3}) = 3\cos 3\frac{\pi}{3} = 3\cos\pi = 3(-1) < 0$

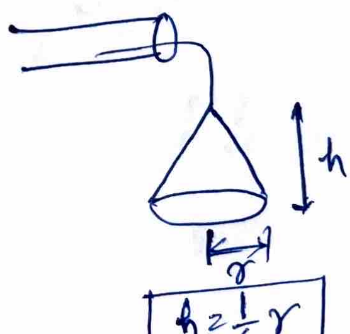
$\therefore$ In the interval $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ the function is decreasing.

interval	Nature of f^n	Sign of f^n
$[0, \pi/6)$	Increasing	+ve
$\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$	Decreasing.	−ve

VVI

Q.6. Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm.

Sol^n. Given, $\frac{dv}{dt} = 12\ cm^3/s$.



$\boxed{h = \frac{1}{6} r}$

A/Q. Let hieght of the cone = h

radius " " " = r

$\therefore h = \frac{1}{6} r$

$\therefore \left.\frac{dh}{dt}\right|_{h = 4\,cm} = ?$

We know that

Volume of the cone $\Rightarrow V = \frac{1}{3}\pi r^2 h \ldots$ i

$\Rightarrow V = \frac{1}{3}\pi (6h)^2 h$

$\Rightarrow V = \frac{1}{3}\pi\, 36\, h^3$

$\Rightarrow V = 12\pi h^3$

Differentiating both side w.r.t t

$\frac{dv}{dt} = 12\pi\, 3h^2 \frac{dh}{dt}$

$\Rightarrow \cancel{12} = \cancel{12}\pi \times 3h^2 \frac{dh}{dt}$

$\Rightarrow \frac{dh}{dt} = \frac{1}{3\pi h^2}$

$\Rightarrow \left.\frac{dh}{dt}\right|_{h=4cm} = \frac{1}{3\pi \times 4^2}$

$= \frac{1}{48\pi}$ cm/sec.

Q. 7. Find the all points of local maxima and local minima of the function f given by,

Sol^n $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$

$f'(x) = 12x^3 + 12x^2 - 24x = 0$

$\Rightarrow 12x(x^2 + x - 2) = 0$

$\Rightarrow x(x^2 + 2x - x - 2) = 0$

$\Rightarrow x[x(x+2) - 1(x+2)] = 0$

$\Rightarrow x(x+2)(x-1) = 0$

$\therefore x = 0, 1, -2$ [critical points]

$f''(x) = 36x^2 + 24x - 24$

Now, at $x = -2$,

$f''(-2) = 36(-2)^2 + 24(-2) - 24 = 144 - 48 - 24$
$= 72 > 0$

$\therefore x = -2$ is the point of local minima.

$\therefore$ local minimum value

$f(-2) = 3(-2)^4 + 4(-2)^3 - 12(-2)^2 + 12$
$= 48 - 32 + 24 + 12 = 84 - 32 = 52$

$\boxed{f(-2) = 52}$

At $x = 1$,

$f''(1) = 36(1)^2 + \cancel{24(1)} - \cancel{24} = 36 > 0$

$x = 1$ is also point of local minima.

$\therefore$ local minimum value

$\therefore f(1) = 3(1)^4 + 4(1)^3 - \cancel{12}(1)^2 \cancel{+12}$

$\boxed{f(1) = 7}$

At $x = 0$,

$f''(0) = -24 < 0$

$\therefore x = 0$ is the point of local maxima

$\therefore$ local maximum value.

$\boxed{f(0) = 12}$

Q.8. Prove that the function $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$ is an increasing function of x throughout its domain.

Sol^n Given function,

$$y = \log(1+x) - \frac{2x}{2+x}, \quad x > -1$$

Differentiating both side w.r.t x.

$$\frac{dy}{dx} = \frac{1}{1+x} - \frac{2(2+x) - 1(2x)}{(2+x)^2}$$

$$= \frac{1}{1+x} - \frac{(4 + \cancel{2x} - \cancel{2x})}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{\cancel{4} + \cancel{4x} + x^2 - \cancel{4} - \cancel{4x}}{(1+x)(2+x)^2}$$

$$\frac{dy}{dx} = \frac{x^2}{(1+x)(2+x)^2}$$

Here, x^2 and $(2+x)^2$ is always positive.

and the term $1+x$ is always positive for any $x > -1$

$\therefore$ overall $\frac{dy}{dx} = \frac{x^2}{(1+x)(2+x)^2} > 0$

$\therefore$ the function $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$ is increasing function of x throughout its domain.

Q.9. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-co-ordinate is changing 8 times as fast as the x-cordinate.

Soln The given curve

$$6y = x^3 + 2 \quad \text{—— (1)}$$

A/Q. $dy = 8\,dx$

$$\Rightarrow \frac{dy}{dx} = 8$$

Differentiate the function w.r.t x

$$6\frac{dy}{dx} = 3x^2$$

$\cancel{6}^{2} \times 8 = \cancel{3}x^2$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

Taking $x = 4$

① $\Rightarrow 6y = 4^3 + 2$

$\Rightarrow 6y = 66$

$\Rightarrow y = 11$

Taking $x = -4$

① $\Rightarrow 6y = (-4)^3 + 2$

$\Rightarrow 6y = -64 + 2$

$\Rightarrow y = \frac{-62}{6} = -31/3$

∴ The points on the curve

$(4, 11)$ or $(-4, -31/3)$.

Q.(10) Show that the function $f(x) = \cos 3x$ is neither strictly increasing nor decreasing on $(0, \frac{\pi}{2})$.

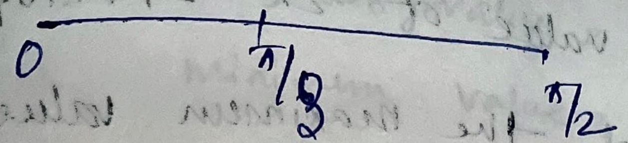
Sol^n Given function is $f(x) = \cos 3x$

$\Rightarrow f'(x) = -3\sin 3x = 0$

$\Rightarrow \sin 3x = 0$

$\Rightarrow \sin 3x = \sin 0 \text{ or } \sin \pi$

$3x = \pi \Rightarrow \boxed{x = \frac{\pi}{3}}$ (Critical pt)



in the interval $\left(0, \frac{\pi}{3}\right)$, let $x = \pi/12$

$$f'(\pi/12) = -3 \sin 3 \frac{\pi}{12}$$
$$= -3 \sin \frac{\pi}{4}.$$
$$= -3 \frac{1}{\sqrt{2}} < 0$$

∴ In interval $\left(0, \frac{\pi}{3}\right)$ the function is strictly decreasing.

Again, In the interval $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. let $x = \frac{5\pi}{12}$.

$$\therefore f'\left(\frac{5\pi}{12}\right) = -3 \sin 3 \frac{5\pi}{12}$$
$$= -3 \sin \frac{5\pi}{4}$$
$$= -3 \sin\left(\pi + \frac{\pi}{4}\right)$$
$$= -3\left(-\sin \frac{\pi}{4}\right)$$
$$= 3 \sin \frac{\pi}{4}$$
$$= 3 \frac{1}{\sqrt{2}} > 0$$

∴ In $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$ the function is strictly increasing

∴ in the interval $\left(0, \frac{\pi}{2}\right)$ the function is ~~height~~ neither increasing nor decreasing.

(Q.4) Find the maximum value of $2x^3-24x+107$ in the intererval $[1,3]$. Find the maximum value of the same function in $[-3,-1]$

Sol^n

$f(x) = 2x^3 - 24x + 107$

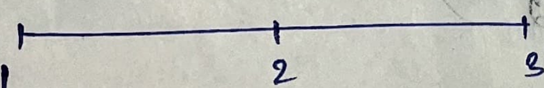
$f'(x) = 6x^2 - 24 = 0$

$\Rightarrow 6x^2 = 24$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$ [critical point]

For the intererval. $[1,3]$



$f''(x) = 12x$

$f''(1) = 12 \times 1 = 12$

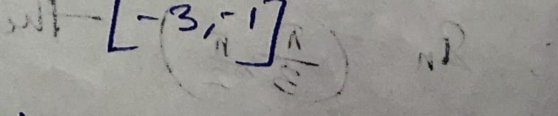
$f''(2) = 12 \times 2 = 24$

$f''(3) = 12 \times 3 = 36$

$\therefore$ maximum value $= 36$ at $x = 3$

minimum value $= 12$ at $x = 1$

Again in the intererval $[-3,-1]$



$f''(-3) = 12(-3) = -36$

$f''(-2) = 12(-2) = -24$

$f''(-1) = 12(-1) = -12$

$\therefore$ maximum value $= -12$ at $x = -1$

minimum value $= -36$ at $x = -3$,